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PLANE TRIGONOMETRY

PLANE TRIGONOMETRY

A TEXTBOOK FOR SCHOOLS AND COLLEGES

BY

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PREFACE

The special features of this book, which are described below, originate in a certain underlying purpose. This is to keep clearly before the student the values which he may expect to receive in return for his effort. An understanding of this sort is necessary for intelligent work. And, if he is to work with enthusiasm, the modern student is inclined to insist on it. Certainly this is not a new idea in the writing of textbooks. Nevertheless the technical methods by which this principle is carried out form the feature that most distinguishes this book from other books in the same field and that motivates most of the other features described below. Whether the reward for labor be immediate and obvious, or whether it be more remote, by example and by explanation it is held before the student as a guide and an incentive. This principle is present from the first problem which sets forth a reason for the introduction of the trigonometric functions before the functions themselves are defined, to the more algebraic parts of the book where clear explanations are given for the methods employed.

1. Originating in this general idea is the **discussion of the significance of numerical data,** and of the criteria for determining the accuracy of results. This material gives the student a sense of the reality of the things with which he is dealing and of the validity of his results. It is a feature of the book most valuable both for its direct applications and for its general cultural benefit.

2. In addition to formal exercises there is a **great variety of applied problems,** an unusual number of which have some real interest to the student and correspond to conditions that he can conceive of as occurring in actual experience.

3. Throughout the book there is an **immediate application of principles to problems.** These take the form both of illustrative examples worked out in the text and of exercises for the student.

A comparison of this book with other texts will show that to an unusual degree the passages devoted to the accumulation of principles and formulas are broken into short sections followed by exercises. This arrangement, besides immediately revealing the usefulness of a method just learned, enables a class to make continuous daily progress in the theory and at the same time to have ample time and material for drill work. Noticeable instances of this method will be found in the work on the logarithmic solution of right triangles, the proofs of identities involving multiple angles, and the formulas for the solution of the oblique triangle.

4. While no formal analysis is given of the various types of **trigonometric identities and equations,** general suggestions are offered which are very helpful in this most puzzling portion of the course. Sufficient justification is given for the introduction of this work and for the methods on which the book insists to dispel the gloomy indifference with which a class frequently struggles with this material.

5. The book may be said to be **self-contained.** (i) It contains a treatment of logarithms. (ii) It defines the terms and explains the principles involved in applied problems, with a saving of time for the class and teacher. (iii) It gives such abundant drill exercises as to make other exercises unnecessary, the problem lists and illustrative examples being complete even for such mechanical work as the use of tables. (iv) The text and tables are designed to be used together, so that the units involved and the methods of interpolation will correspond.

6. The book is especially **adaptable to courses of various lengths and purposes.** Certain sections are starred for omission in short courses. Exercises are given in duplicate with answers supplied where practicable for the odd-numbered examples. Either the odd- or even-numbered examples are more than sufficient for an ordinary class. The exercises are so carefully graded that a simple course may be planned, using only the earlier exercises in each section. Many types of applied problems are segregated so that certain types may be omitted to suit the time or purpose of a class.

7. It has been intended to make the book **concise but sufficiently complete,** and to include only such material as is justified by its true value rather than by tradition. No attempt has been made to reduce it to a mere outline, which, in the end, is wasteful of time and unsatisfying to the student.

8. While the book is modern in its purpose and material, it is **conservative in the details of presentation.** It avoids peculiar methods which may be useful for some individual instructor but are dangerous in the hands of an inexperienced teacher and offensive or inconvenient to an experienced one. On the contrary, the book should prove helpful to an inexperienced teacher by revealing the purpose behind each detail, and should lend itself to the methods of any teacher who has already developed his own technique of presentation, by furnishing ample material with which his class may work.

The author wishes to express his gratitude to Professor Leon Archibald for drawing the figures, and to Doctor Elizabeth Carlson for suggestions and problems. He is especially grateful to Professor Anthony L. Underhill who read the entire manuscript, used parts of it in the classroom, and made innumerable criticisms and suggestions which have added immensely to the value of the book.

R. W. B.

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PLANE TRIGONOMETRY

GREEK ALPHABET

A, α — Alpha	I, ι — Iota	P, ρ — Rho
B, β — Beta	K, κ — Kappa	Σ, σ, ς — Sigma
Γ, γ — Gamma	Λ, λ — Lambda	T, τ — Tau
Δ, δ — Delta	M, μ — Mu	Υ, υ — Upsilon
E, ϵ — Epsilon	N, ν — Nu	Φ, ϕ — Phi
Z, ζ — Zeta	Ξ, ξ — Xi	X, χ — Chi
H, η — Eta	O, o — Omicron	Ψ, ψ — Psi
Θ, θ — Theta	Π, π — Pi	Ω, ω — Omega

PLANE TRIGONOMETRY

CHAPTER I

THE TRIGONOMETRIC FUNCTIONS*

1. A problem of measurement and a graphical solution. — It is desired to build a bridge across a certain river, and in order to do so it is necessary to find the width of the river. Instruments are available, of course, for measuring angles and distances on land with the necessary accuracy. The problem is to devise a method by which, from such measurements, one can find the distance across the stream, where no direct measurement is possible.

To carry out the solution, place marks at two points, A and B, on opposite banks of the stream. With an instrument placed at A, establish a line AM at right angles to AB, and along AM measure any convenient distance, AC. With an instrument at C then measure the angle ACB. Suppose that in the present case the distance AC is taken equal to 800 feet, and that, on measurement, the angle ACB is found to be equal to 71°. The distance AB can now be found graphically as follows: Construct on paper a triangle $A'B'C'$ similar to ABC, making the angle at A' a right angle, and with

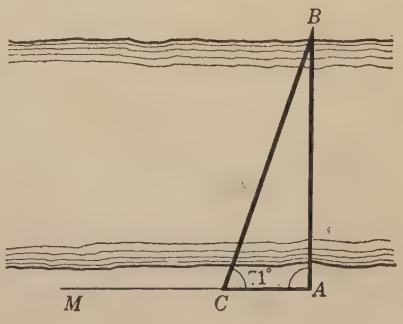

FIG. 1

* After reading Articles 1 and 2, the reader may proceed at once to Chapter II, and complete Chapter I after reading Chapters II, III, and IV. This order is recommended for those wishing a very short course.

a protractor make the angle at C' equal to 71°. The side $A'B'$ can be measured. And since the triangles are similar, AB can be found from the proportion

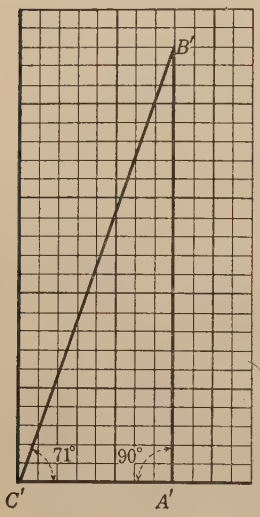

Fig. 2

(1) $AB : AC = A'B' : A'C'$

where AB is the only unknown.

For convenience use paper ruled into squares and allow one space on the paper to represent some distance, say 100 feet, of the problem. Then $A'C'$, which corresponds to 800 feet, is 8 spaces. We read off the distance $A'B'$, which is about 23.2 spaces, and conclude that, approximately,

(2) $AB = 2320$ feet.

The accuracy of this result is limited by errors made in measuring the angles at A and C and the distance AC, and also, probably to even a greater extent, by inaccuracies in constructing the angle at C' and reading the distance $A'B'$.

EXERCISES

Using Figure 2, determine AB for each of the following measurements of AC, assuming that angle $ACB = 71°$.

1. $AC = 1000$ ft. *Ans.* $AB = 2900$ ft. **2.** $AC = 1000$ yds.
3. $AC = 300$ rds. *Ans.* $AB = 870$ rds. **4.** $AC = 30$ rds.

Using Figure 3, determine AB for each of the following pairs of measurements. Do not draw lines nor mark on the figure, but indicate angles by a celluloid ruler or a thread stretched through the center of the circle.

5. $AC = 1000$ ft., $\angle ACB = 61°$. *Ans.* 1800 ft.
6. $AC = 700$ ft., $\angle ACB = 58°$.
7. $AC = 1100$ ft., $\angle ACB = 42°$. *Ans.* 990 ft.
8. $AC = 900$ ft., $\angle ACB = 39°$.

9. $AC =$ 800 ft., $\angle ACB = 52° 30'$. *Ans.* 1040 ft.

10. $AC =$ 650 ft., $\angle ACB = 31°$.

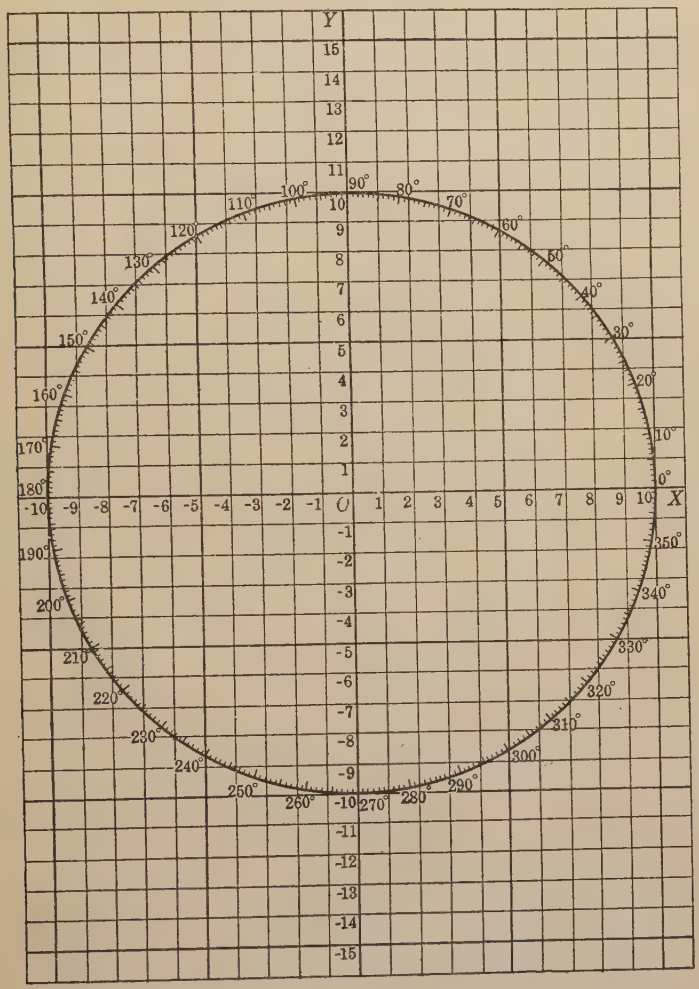

FIG. 3

2. Another solution. — From equation (2), Article **1**, we have:

$$\frac{AB}{AC} = \frac{2320}{800} = 2.9.$$

From the results of Exercise 1, Article **1**, we find

$$\frac{AB}{AC} = \frac{2900}{1000} = 2.9,$$

and from Exercise 3,

$$\frac{AB}{AC} = \frac{870}{300} = 2.9.$$

The ratio AB/AC evidently retains the value 2.9, regardless of the length of AC. Indeed, since the right triangles are similar, this ratio must remain constant so long as the angle ACB remains equal to 71°. A change in the value of the angle ACB carries with it a change in the ratio AB/AC. This ratio, then, is a number which is independent of the dimensions of the triangle, and is determined solely by the magnitude of the angle ACB. The ratio AB/AC, since it depends for its value on the value of the angle, is said to be a *function* of the angle ACB.

This function is called *the tangent of the angle ACB*, a statement which we may write in the following form:

(1) $$\frac{AB}{AC} = \tan \angle ACB.$$

A general definition of the tangent of an angle is given in Article **8**. We have seen that when $\angle ACB = 71°$, $AB/AC = 2.9$, so that $\tan 71° = 2.9$.

Had we but known this value for tan 71°, we could have solved the problem of Article **1**, without incurring the inaccuracy and inconvenience of the graphical method. For, since $AC = 800$ ft., by equation (1)

$$AB = 800 \times \tan 71° = 800 \times 2.9 = 2320 \text{ ft.}$$

The solution which uses the tangent of the angle ACB is both more accurate and more convenient than the graphical method, for tables have been made which give very exactly the value of the tangent of any angle.

EXERCISES

1. A table gives tan 71° = 2.9042. Use this value to obtain a better answer to the problem of Article **1**. *Ans.* 2323 ft.

2. Solve Exercises **5** to **10**, Article **1**, using the following values:

tan 31° = .601, tan 39° = .810,
tan 42° = .9004, tan 52° 30′ = 1.303,
tan 58° = 1.600, tan 61° = 1.804.

Ans. (**5**) 1804 ft., (**7**) 990 ft., (**9**) 1042 ft.

Using Figure 3, find the values of

3. tan 27° 30′. *Ans.* .52. **4.** tan 11°.
5. tan 55°. *Ans.* 1.43. **6.** tan 45°.
7. tan 60°. *Ans.* 1.7. **8.** tan 75°.
9. tan 32°. *Ans.* .62. **10.** tan 25°.

Using Figure 3, find an angle C, such that

11. tan C = 1.1. *Ans.* C = 48°. **12.** tan C = .51.
13. tan C = .65. *Ans.* C = 33°. **14.** tan C = .36.
15. tan C = .70. *Ans.* C = 35°. **16.** tan C = .23.
17. tan C = 4.0. *Ans.* C = 76°. **18.** tan C = .87.

3. Directed distances.* — The next five articles deal with topics necessary for understanding a general definition of the tangent and of certain other functions of an angle.

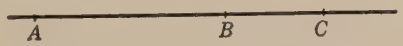

FIG. 4

If a point moves in a straight line from a point A to a point C, it traces out the *line-segment* AC. We call A the *initial point* and C the

* See footnote, page 1.

terminal point of the segment. We also denote by AC the distance from A to C or the length of AC expressed in some chosen unit. If B is a point on the line between A and C, in tracing AC the moving point traces the segments AB and BC, and

$$AB + BC = AC.$$

In order that this rule may hold even when B is not between A and C, distances measured in one direction along any given line will be counted positive, and distances in the opposite direction will be counted negative. That is, $AB = -BA$, or

$$AB + BA = 0. \tag{1}$$

If a point starts at A and, after moving back and forth along the line in any way, ultimately arrives at C, the algebraic sum of all the distances traversed by the point is AC. For every segment except AC is traversed the same number of times in each direction, and the sum of the lengths of these other segments is zero. In particular, if the tracing point moves from A to B and then to C, we have

$$AB + BC = AC, \tag{2}$$

for all arrangements of the points on a straight line.

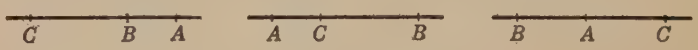

Fig. 5

4. The coördinate system. — Consider two straight lines drawn at right angles to each other in a plane. One of them, OX, usually taken horizontally, we shall call the x-axis, while the vertical line, OY, will be called the y-axis. Their point of intersection, O, is the *origin*. We choose some unit of length in which to express distances, and agree that distances measured along any line which is parallel to OX are to be counted positive if measured to the right, and negative if measured to the left. Distances along any line parallel to OY are positive if measured upward, negative if measured downward.

Now let P be any point of the plane, and let its distance from OY be x and its distance from OX be y. Then x is said to be the *abscissa* of the point P, and y is the *ordinate* of P. The position of P is determined if these two numbers or *coördinates*, x and y, are known, and P is commonly denoted as the point (x, y). The points $(-4, 6)$, $(-6, -5)$ and $(7, -3)$ are marked in Figure 6.

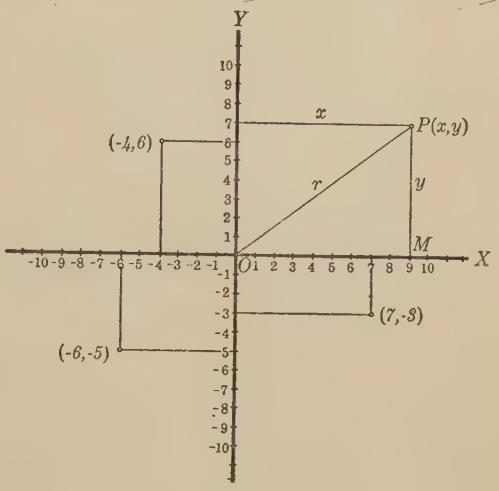

FIG. 6

The distance of the point P from the origin is called the *radius vector* of P. This distance r is counted positive, regardless of the signs of x and y. From the triangle OMP, $r = \sqrt{x^2 + y^2}$.

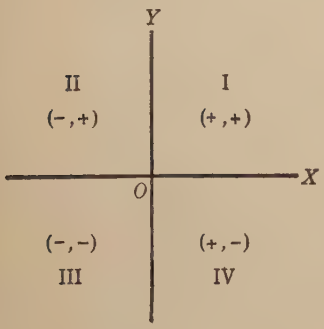

FIG. 7

The axes of coördinates divide the plane into four *quadrants* which are numbered as in Figure 7. This figure also shows the signs of the coördinates of points in the different quadrants.

EXERCISES

For careful plotting, the student should use printed "cross-section" or "coördinate" paper. For hurried work, or at the blackboard, mark a scale of points on each axis, and then estimate distances by the eye or with a measure.

1. Plot the points: $(2, 1), (4, 7), (6, 6), (-2, 4), (-5, 2), (-4, -3), (-5, -5), (4, -1), (2, -8), (\sqrt{5}, 3), (-1, -\sqrt{3}), (\pi, 4), (2, \pi), (0, 3), (0, 0), (6, 0), (-3, 0), (0, -4), (7/2, 2), (-2/3, 5/2)$.

2. In Figure 3, give the coördinates for the point at which the degree sign (°) is printed for each of the angles 0°, 10°, 20°, etc.

Using Figure 3, find the value of the angle XOP to the nearest degree for each of the following positions of P.

3. $(10, 4)$.	*Ans.* 22°.	**4.** $(8, 7)$.
5. $(4, 8)$.	*Ans.* 63°.	**6.** $(2, 4)$.
7. $(13, 0)$.	*Ans.* 0°.	**8.** $(0, 5)$.
9. $(-3, 10)$.	*Ans.* 107°.	**10.** $(-12, 5)$.
11. $(-9, 3)$.	*Ans.* 162°.	**12.** $(-12, 4)$.
13. $(-6, 2)$.	*Ans.* 162°.	**14.** $(-3/2, 12)$.

Denote by A the foot of the perpendicular drawn from a point B to the x-axis, and find the value of tan $\angle AOB$ for each of the following positions of B.

15. $(4, 12)$.	*Ans.* $\tan \angle AOB = 3$.	**16.** $(3, 9)$.
17. $(4, 5)$.	*Ans.* $\tan \angle AOB = 5/4$.	**18.** $(3, 3)$.
19. $(\pi, 1)$.	*Ans.* $\tan \angle AOB = 1/\pi$.	**20.** $(5, 10)$.

Find the radius vector of each of the following points.

21. $(3, 4)$.	*Ans.* 5.	**22.** $(12, 5)$.
23. $(15, 8)$.	*Ans.* 17.	**24.** $(2, 1)$.
25. $(5, -12)$.	*Ans.* 13.	**26.** $(-4, 3)$.
27. $(-1, 1)$.	*Ans.* $\sqrt{2}$.	**28.** (x, y).

5. The generation of angles. — If a straight line, l, revolves in a plane about a point O from the position OA to the position OC, it sweeps out or generates the *angle* AOC. The lines OA and OC are called the *sides* of the angle AOC, OA being the *initial side*, and OC being the *terminal side* of the angle.

When there is no possibility of confusion, an angle may be denoted by the same single letter as its vertex, or by some different letter (usually a Greek letter) designated for the purpose. For example, we may say

$$\angle AOC = \angle O = \theta.$$

If OB is a line lying between OA and OC, in revolving from the position OA to OC, the generating line sweeps out first the angle AOB, then the angle BOC, and in all, the angle AOC. We can thus add angles and write

$$\angle AOB + \angle BOC = \angle AOC.$$

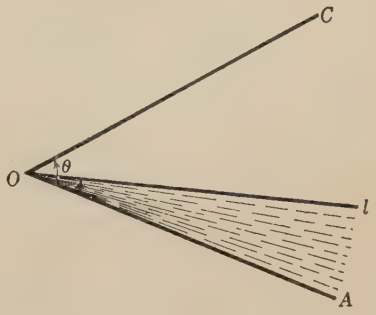

Fig. 8

6. General and directed angles. — It has been customary in elementary geometry to consider all angles as positive and as limited to values less than four right angles. These restrictions are not necessary and it is now desirable to do away with them.

We shall call certain angles positive and others negative, the sign of an angle depending on the direction of rotation of the line which generated it. An angle which is generated by a line revolving in a *counter-clockwise* sense (that is, in the opposite direction from that in which a hand of a clock moves) is counted positive. And an angle generated by a line turning in a clockwise direction is negative.

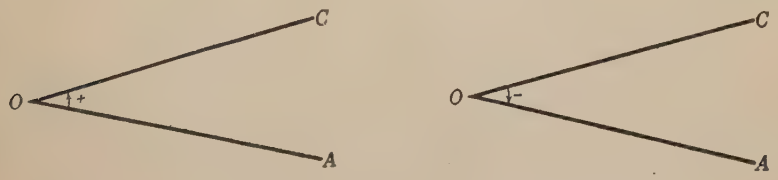

Fig. 9

In Figure 9, $\angle AOC$ is positive and $\angle COA$ is negative. In fact

$$\angle COA = -\angle AOC.$$

We shall place no limit on the magnitude of an angle. During one complete counter-clockwise revolution a line sweeps out all angles from 0° to 360°. But it may continue to turn. During its second revolution it completes the generation of angles between 360° and 720°; during the next revolution it finishes generating the angles from 720° to 1080°, etc., without arbitrary limit.

We thus see that two lines OA and OC may serve as the initial and terminal sides respectively not merely of a single angle AOC, but of countless angles, some positive and some negative.

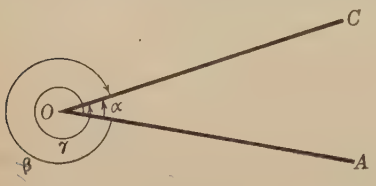

FIG. 10

In Figure 10, for example, the angle AOC may have any one of the values $\alpha = 30°$, $\beta = -330°$, $\gamma = 390°$, etc.

The value of an angle AOC therefore depends not only on the positions of the initial side OA and of the terminal side OC, but also on the direction and extent of rotation of the generating line. Any two values of AOC will differ by some multiple of 360°.

Thus, in the figure, $\alpha - \gamma = 30° - 390° = -360°$; $\alpha - \beta = 30° - (-330°) = 360°$; $\gamma - \beta = 390° - (-330°) = 2 \times 360°$.

EXERCISES

Draw the following angles, and indicate by arrows, as in Figure 10, the direction and extent of the rotation. In each case name two other angles, one positive and one negative, having the same initial and terminal sides.

1. 90°.	**2.** −90°.	**3.** 135°.
4. −135°.	**5.** 180°.	**6.** −180°.
7. 270°.	**8.** −270°.	**9.** 300°.
10. 360°.	**11.** −390°.	**12.** 450°.
13. −600°.	**14.** 1035°.	**15.** −120°.

7. The standard position of an angle. — An angle is said to be in *standard position* with reference to a system of coördinates, if its vertex is at the origin and if its initial side extends along the positive x-axis. We speak of an angle as being in, say, the second quadrant, if, when it is placed in standard position, its terminal side lies in the second quadrant. We use similar expressions, of course, for the other quadrants.

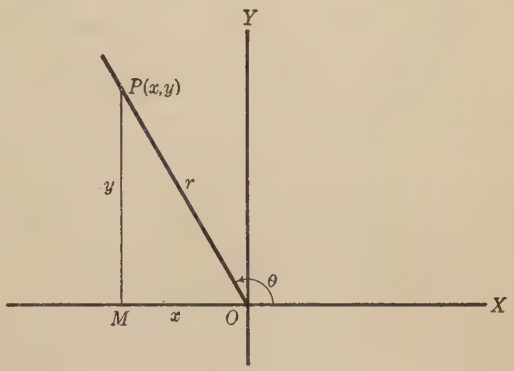

FIG. 11

The angles 120°, −210°, and 510°, for example, are in the second quadrant, while 65° is in the first, −100° is in the third, and 300° is in the fourth quadrant. Positive angles between 0° and 90° are in the first quadrant; angles between 90° and 180° are in the second, angles between 180° and 270° are in the third, and angles between 270° and 360° are in the fourth quadrant.

EXERCISES

Using Figure 3, find one positive and one negative value of the angle XOP, for each of the following positions of P, giving the results correctly to the nearest degree.

1. $(-7, 7)$.	*Ans.* 135°, −225°.	**2.** $(-6, 6)$.
3. $(5, -7)$.	*Ans.* 306°, −54°.	**4.** $(10, -3)$.
5. $(8, 0)$.	*Ans.* 360°, −360°.	**6.** $(0, -3)$.
7. $(-7, -10)$.	*Ans.* 235°, −125°.	**8.** $(8, 1)$.

8. Definition of the trigonometric functions of an angle. — In Article **2** we defined, for any *acute* angle, a number which we called its tangent. Now we shall give definitions of the tangent and of

five other functions of an angle which will be applicable to *any* angle, θ, whatsoever. These six quantities, called trigonometric functions of θ, are the sine, the cosine, the tangent, the cotangent, the secant, and the cosecant of θ, which are written $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$, respectively. *Definition: Place the angle θ in standard position with reference to a coördinate system, and choose any point, P, on the terminal side of θ. Let the coördinates of this point be (x, y) and its distance from the origin be r. Then*

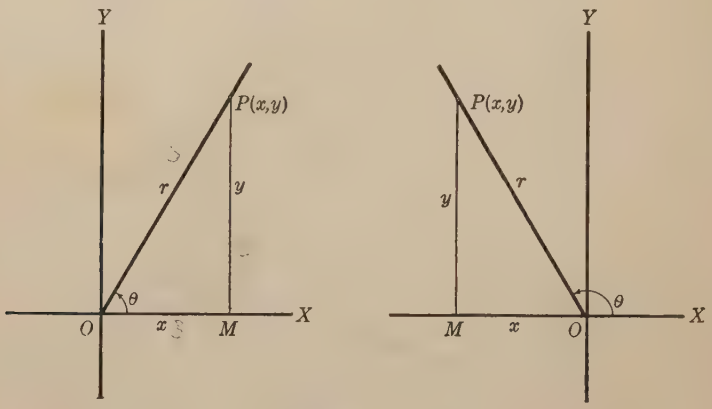

FIG. 12

$$\sin\theta = \frac{\text{ordinate of } P}{\text{radius vector of } P} = \frac{y}{r},$$

$$\cos\theta = \frac{\text{abscissa of } P}{\text{radius vector of } P} = \frac{x}{r},$$

$$\tan\theta = \frac{\text{ordinate of } P}{\text{abscissa of } P} = \frac{y}{x},$$

$$\cot\theta = \frac{\text{abscissa of } P}{\text{ordinate of } P} = \frac{x}{y},$$

$$\sec\theta = \frac{\text{radius vector of } P}{\text{abscissa of } P} = \frac{r}{x},$$

$$\csc\theta = \frac{\text{radius vector of } P}{\text{ordinate of } P} = \frac{r}{y}.$$

These quantities,* although defined in terms of the coördinates of the point P, are nevertheless true functions of θ. That is, their values depend solely on the value of θ, and not on the particular point P which we choose on the terminal side of θ. For if we choose any other point, P', on OP, with coördinates (x', y') and radius vector r', from the similarity of the triangles OMP and $OM'P'$, we have

$$\frac{y'}{r'} = \frac{y}{r},$$

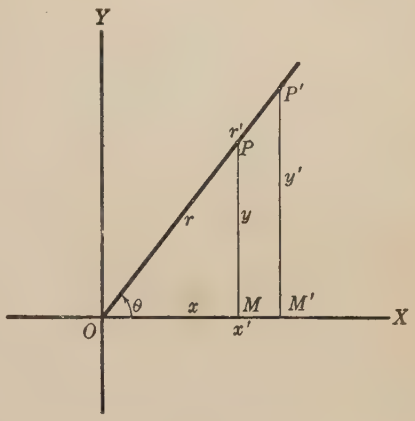

Fig. 13

so that the value of sin θ is the same whether we obtain it using P' or using P. The student should write the corresponding equations for the other functions.

Two numbers are said to be *reciprocals* each of the other, if their product is equal to 1. Thus 3 and 1/3, and 2/5 and 5/2 are pairs of reciprocals. If a and b are reciprocals $a = 1/b$.

Since, by definition, $\sin\theta = y/r$ and $\csc\theta = r/y$, these functions are reciprocals each of the other. That is,

$$\csc\theta = \frac{1}{\sin\theta}.$$

* Two other functions are sometimes used, particularly in laying out railroad curves. They are the *versed sine* of θ and the *coversed sine* of θ, defined as

$$\textit{vers}\,\theta = 1 - \cos\theta,$$
$$\textit{covers}\,\theta = 1 - \sin\theta.$$

We shall make no further reference to these functions.

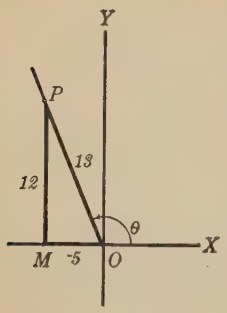

FIG. 14

Likewise $\sec\theta = \dfrac{1}{\cos\theta}$,

and $\cot\theta = \dfrac{1}{\tan\theta}$.

EXAMPLE 1. Find the trigonometric functions of the angle XOP, if the coördinates of P are $(-5,12)$.

The radius vector is

$$r = \sqrt{x^2 + y^2} = \sqrt{(-5)^2 + (12)^2} = 13.$$

Therefore, by definition,

$$\sin\theta = \frac{y}{r} = \frac{12}{13} = .923, \qquad \cos\theta = \frac{x}{r} = \frac{-5}{13} = -.385,$$

$$\tan\theta = \frac{y}{x} = \frac{12}{-5} = -\frac{12}{5} = -2.40, \qquad \cot\theta = \frac{x}{y} = \frac{-5}{12} = -.417,$$

$$\sec\theta = \frac{r}{x} = -\frac{13}{5} = -2.60, \qquad \csc\theta = \frac{r}{y} = \frac{13}{12} = 1.083.$$

The last three values might have been obtained from the first three by the reciprocal relations. That is, since

$$\sin\theta = 12/13, \qquad \csc\theta = 13/12;$$

since $\cos\theta = -5/13, \qquad \sec\theta = -13/5;$

since $\tan\theta = -12/5, \qquad \cot\theta = -5/12.$

EXAMPLE 2. Find the trigonometric functions of 270°.

Place the angle in standard position, and as P choose a convenient point, say $(0,-8)$. Then $r = +8$, since r is always counted positive. Then, from the definitions,

$$\sin 270° = \frac{y}{r} = \frac{-8}{8} = -1, \qquad \cos 270° = \frac{x}{r} = \frac{0}{8} = 0.$$

If we try to find $\tan\theta$, we are tempted to write

$$\tan\theta = \frac{y}{x} = \frac{-8}{0};$$

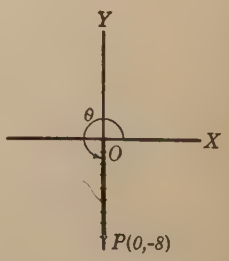

FIG. 15

but, since division by zero has no meaning, evidently there is no such thing as tan 270°. Likewise sec 270° does not exist, since r/x or $8/0$ is without meaning. But

$$\cot 270° = \frac{x}{y} = \frac{0}{-8} = 0, \qquad \csc 270° = \frac{r}{y} = \frac{8}{-8} = -1.$$

EXAMPLE 3. Find the trigonometric functions of 214°. From Figure 3, we see that the point $(-9,-6)$ lies on the terminal side of the angle 214°. Let us use $(-9,-6)$ as the point P. Then

$$r = \sqrt{(-9)^2 + (-6)^2} = \sqrt{117} = 10.8.$$

Therefore

$$\sin 214° = \frac{y}{r} = \frac{-6}{10.8} = -.56, \qquad \cos 214° = \frac{x}{r} = \frac{-9}{10.8} = -.83,$$

$$\tan 214° = \frac{y}{x} = \frac{-6}{-9} = .67, \qquad \cot 214° = \frac{x}{y} = \frac{-9}{-6} = 1.5,$$

$$\sec 214° = \frac{r}{x} = \frac{10.8}{-9} = -1.2, \qquad \csc 214° = \frac{r}{y} = \frac{10.8}{-6} = -1.8.$$

Second solution. — In Figure 3, the radius of the circle is 10. As the point P, take the point on the circle corresponding to 214°. Its coördinates are $(-8.3,-5.6)$ and $r = 10$. Therefore $\sin 214° = y/r = -.56$, $\cos 214° = -.83$. To find the tangent take P as the point where the terminal side of 214° meets the line for which $x = -10$, the vertical tangent to the circle. Its coördinates are $(-10,-6.7)$. Therefore $\tan 214° = y/x = .67$. If we take P on the horizontal tangent to the circle, its coördinates are $(-15,-10)$, and $\cot\theta = x/y = 1.5$.

EXERCISES

ORAL EXERCISES

Find the value of r for each of the following positions of P, and then find the trigonometric functions of the angle XOP. Do not reduce the answers to decimal form.

1. $(3,-4)$.

Ans. $r = 5$, $\sin\theta = -4/5$, $\cos\theta = 3/5$, $\tan\theta = -4/3$, etc.

2. $(-8, 6)$. **3.** $(2, 3)$. **4.** $(-4,-3)$.

5. $(5, 12)$. **6.** $(-12, 5)$. **7.** $(-5,-12)$.

8. $(1, 1)$. **9.** $(-1, 1)$. **10.** $(1,-1)$.

11. (2, 2). **12.** (−2, 2). **13.** (2,−2).
14. (−3, 4). **15.** (−3,−4). **16.** (−12,−5).
17. (6, 8). **18.** (−6, 8). **19.** (−6,−8).
20. (0, 2). **21.** (2, 0). **22.** (−3, 0).

23. For which quadrants is

(*a*) sin θ positive; (*b*) sin θ negative;
(*c*) cos θ positive; (*d*) cos θ negative;
(*e*) tan θ positive; (*f*) cot θ negative;
(*g*) sec θ positive; (*h*) csc θ positive?

24. The signs of sin θ and csc θ are always the same. (*a*) Why is this true? (*b*) What other function always has the same sign as cos θ? (*c*) What function has the same sign as tan θ?

25. In what quadrant is θ in each of the following cases?

(*a*) sin θ and cos θ are both positive.
(*b*) sin θ is positive, cos θ is negative.
(*c*) sin θ and cos θ are negative.
(*d*) sin θ is negative, cos θ is positive.

26. Work Exercise **25** using tan θ first instead of cos θ and then instead of sin θ.

WRITTEN EXERCISES

Using Figure 3, find the values of the sine, cosine, tangent and cotangent of the following angles correctly to one figure in every case and correctly to two figures when possible.

27. 49°. *Ans.* .75, .66, 1.2, .87. **28.** 65°.
29. 29°. *Ans.* .48, .87, .55, 1.8. **30.** 14°.
31. 148°. *Ans.* .53, −.85, −.62, −1.6. **32.** 165°.
33. 240°. *Ans.* −.87, −.50, 1.7, .58. **34.** 209°.
35. 0°. *Ans.* 0, 1, 0, —. **36.** 180°.
37. 90°. *Ans.* 1, 0, —, 0. **38.** −90°.
39. 301°. *Ans.* −.86, .52, −1.7, −.60. **40.** 339°.
41. −50°. *Ans.* −.77, .64, −1.2, −.84. **42.** −69°.
43. 421°. *Ans.* .87, .48, 1.8, .55. **44.** 824°.

9. The determination of one function from another. — If we know the quadrant in which an angle lies, and have the value of one of its trigonometric functions, we can find the values of its other trigonometric functions. Formulas by which this can be done algebraically will be given in Article **68.** The geometrical method is illustrated in the following examples.

EXAMPLE 1. Given that θ is an angle in the second quadrant and that $\tan\theta = -4/5$, find the other functions of θ. Any point (x,y) in the second quadrant is a point on the terminal side of θ, if $y/x = \tan\theta = -4/5$. In particular, the point $(-5,4)$ is such a point. The point $(-10,8)$ or the point $(-5/2,2)$ would serve equally well, but we prefer $(-5,4)$ as being simpler. Then $r = \sqrt{16+25} = \sqrt{41}$. Therefore,

$$\sin\theta = \frac{y}{r} = \frac{4}{\sqrt{41}}, \qquad \cos\theta = \frac{x}{r} = \frac{-5}{\sqrt{41}},$$

$$\cot\theta = \frac{-5}{4}, \qquad \sec\theta = \frac{\sqrt{41}}{-5}, \qquad \csc\theta = \frac{\sqrt{41}}{4}.$$

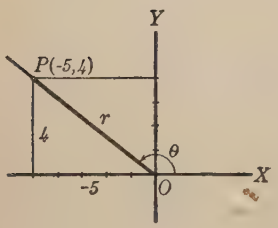

FIG. 16

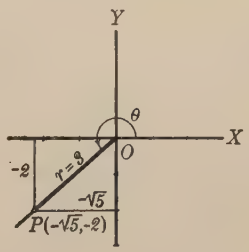

FIG. 17

EXAMPLE 2. Given that $\sin\theta = -2/3$ and that θ is in the third quadrant. Find the other functions of θ. The point in the third quadrant for which $y = -2$ and $r = 3$ must lie on the terminal side of θ. From the relation $x^2 + y^2 = r^2$, $x = -\sqrt{r^2 - y^2} = -\sqrt{9-4} = -\sqrt{5}$. Therefore $\sin\theta = \frac{-2}{3}$, $\cos\theta = \frac{-\sqrt{5}}{3}$, $\tan\theta = \frac{2}{\sqrt{5}} = \frac{2}{5}\sqrt{5}$, $\cot\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = \frac{-3}{\sqrt{5}} = -\frac{3}{5}\sqrt{5}$, $\csc\theta = \frac{-3}{2}$.

In general, the method consists in finding from the given conditions a point on the terminal side of the angle, and then using the co-

ordinates and radius vector of this point in the definitions of the functions of an angle in order to obtain their values.

EXERCISES

In each of the following examples, find the functions of an angle which satisfies the given conditions.

1. $\tan\theta = \frac{2}{3}$, θ in first quadrant.

   Ans. $\frac{2}{13}\sqrt{13}$, $\frac{3}{13}\sqrt{13}$, $\frac{3}{2}$, $\frac{1}{3}\sqrt{13}$, $\frac{1}{2}\sqrt{13}$.

2. $\cot\theta = \frac{4}{3}$, θ in first quadrant.
3. $\sin\theta = .4$, θ in first quadrant.

   Ans. $\frac{1}{5}\sqrt{21}$, $\frac{2}{21}\sqrt{21}$, $\frac{1}{2}\sqrt{21}$, $\frac{5}{21}\sqrt{21}$, $\frac{5}{2}$.

4. $\cos\theta = \frac{6}{11}$, θ in first quadrant.
5. $\sin\theta = \frac{3}{5}$, θ in second quadrant. *Ans.* $-\frac{4}{5}$, $-\frac{3}{4}$, $-\frac{4}{3}$, $-\frac{5}{4}$, $\frac{5}{3}$.
6. $\cos\theta = -\frac{2}{5}$, θ in second quadrant.
7. $\tan\theta = 2$, θ in third quadrant.

   Ans. $-\frac{2}{5}\sqrt{5}$, $-\frac{1}{5}\sqrt{5}$, $\frac{1}{2}$, $-\sqrt{5}$, $-\frac{1}{2}\sqrt{5}$.

8. $\cos\theta = -\frac{1}{2}$, θ in third quadrant.
9. $\sin\theta = -\frac{5}{13}$, θ in fourth quadrant.

   Ans. $\frac{12}{13}$, $-\frac{5}{12}$, $-\frac{12}{5}$, $\frac{13}{12}$, $-\frac{13}{5}$.

10. $\cos\theta = \frac{8}{17}$, θ in fourth quadrant.
11. $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$. *Ans.* $-\frac{4}{3}$, $-\frac{3}{4}$, $-\frac{5}{3}$, $\frac{5}{4}$.
12. $\tan\theta = \frac{8}{15}$, $\cos\theta$ negative.

CHAPTER II

THE TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE*

10. The trigonometric functions of an acute angle. — Consider a right triangle, ABC, in which the right angle is at C. Let α denote the angle at the vertex A, and β the angle at the vertex B. Denote by a, b, and c the lengths of the sides which are opposite the vertices A, B, and C respectively.

Note: This notation will be used regularly. Unless it is otherwise stated, in a triangle whose vertices are the points A, B, C, the letters α, β, γ will denote the angles at those vertices, and a, b, c will denote the lengths of the respectively opposite sides. In a *right* triangle, γ will ordinarily be the right angle.

The six trigonometric functions of the acute angle α are then

$$\sin \alpha = \frac{a}{c}, \qquad \cos \alpha = \frac{b}{c},$$

$$\tan \alpha = \frac{a}{b}, \qquad \cot \alpha = \frac{b}{a},$$

$$\sec \alpha = \frac{c}{b}, \qquad \csc \alpha = \frac{c}{a}.$$

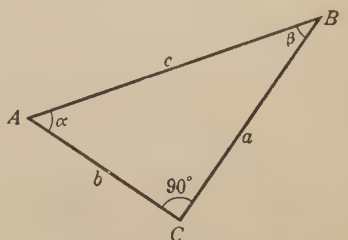

FIG. 18

These statements may be expressed as follows: *For an acute angle of any* ***right*** *triangle,*
the sine equals the ratio of the opposite side to the hypotenuse;
the cosine equals the ratio of the adjacent side to the hypotenuse;
the tangent equals the ratio of the opposite side to the adjacent side;
the cotangent equals the ratio of the adjacent side to the opposite side;
the secant equals the ratio of the hypotenuse to the adjacent side;
the cosecant equals the ratio of the hypotenuse to the opposite side.

* See footnote, page 1.

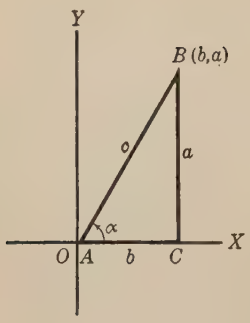

FIG. 19

Students who have not read the latter part of Chapter I may accept the foregoing statements as *definitions* of the trigonometric functions of an acute angle. They are valid for all acute angles, since any acute angle may be taken as one of the angles of a right triangle. Definitions that apply to *all* angles are given in Article **8** of Chapter I.

Students who have learned the general definition of Article **8** should consider the statements above as a *theorem*, which may be proved as follows. Draw a set of coördinate axes with the origin at A and the x-axis in the direction AC, placing angle α in standard position. The coördinates of B are (b,a), and its radius vector is c. Therefore, by Article **8**,

$$\sin \alpha = \frac{a}{c}, \qquad \cos \alpha = \frac{b}{c}, \text{ etc.}$$

The relations of this article should be remembered, also, in the following forms:

$$a = c \sin \alpha, \qquad a = b \tan \alpha,$$
$$b = c \cos \alpha, \qquad b = a \cot \alpha,$$
$$c = a \csc \alpha, \qquad c = b \sec \alpha.$$

11. The variation of the functions of an acute angle. — For a hypotenuse, c, of given length, the larger the angle α, the larger is the value of a and the smaller is the value of b. Therefore, by the relations of Article **10**, when an acute angle increases, its sine, tangent, and secant increase, and its cosine, cotangent, and cosecant decrease. Since a and b are always less than c, the sine and cosine of an acute angle are always less than 1 and the secant and cosecant are always greater than 1. The tangent and cotangent have no such fixed limits.

12. The functions of complementary angles. — If we apply the statements of Article **10** to the angle β instead of to α, since a is now the adjacent side and b is the opposite side, we find that

$$\sin \beta = \frac{b}{c} = \cos \alpha, \qquad \cos \beta = \frac{a}{c} = \sin \alpha,$$

$$\tan\beta = \frac{b}{a} = \cot\alpha, \qquad \cot\beta = \frac{a}{b} = \tan\alpha,$$

$$\sec\beta = \frac{c}{a} = \csc\alpha, \qquad \csc\beta = \frac{c}{b} = \sec\alpha.$$

The sine and cosine are said to be *cofunctions*, each of the other, as are also the tangent and cotangent, and the secant and cosecant. Inasmuch as any two complementary acute angles can be taken as the acute angles, α and β, of a right triangle, we have the

Theorem. Any trigonometric function of a positive acute angle is equal to the corresponding cofunction of the complementary angle.

13. The trigonometric functions of 45°. — It is very easy to compute the values of the trigonometric functions of the angles 30°, 45° and 60°, and it is useful to remember them. In this article we compute the values for 45°.

Draw a right triangle with the acute angle, α, equal to 45°, the length of the opposite side a equal to one unit. Then we also have $\beta = 45°$ and $b = 1$. Since $a^2 + b^2 = c^2$, $c = \sqrt{1+1} = \sqrt{2}$.

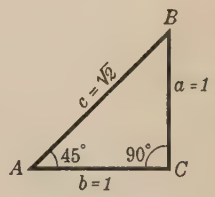

FIG. 20

From Article **10**, we have

$$\sin 45° = \frac{a}{c} = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2}, \qquad \cos 45° = \frac{b}{c} = \tfrac{1}{2}\sqrt{2},$$

$$\tan 45° = \frac{a}{b} = 1, \qquad \cot 45° = \frac{b}{a} = 1,$$

$$\sec 45° = \frac{c}{b} = \frac{\sqrt{2}}{1} = \sqrt{2}, \qquad \csc 45° = \frac{c}{a} = \sqrt{2}.$$

14. The trigonometric functions of 60° and 30°. — Draw an equilateral triangle, ABD, making the length of each side equal to 2 units. From B draw BC perpendicular to AD. It will bisect AD at C. Then in the *right* triangle ABC, $\alpha = 60°$, $\beta = 30°$, $c = 2$, $b = 1$. Moreover $a^2 + b^2 = c^2$, so that $a = \sqrt{c^2 - b^2} = \sqrt{4-1} = \sqrt{3}$.

From Article **10**, we get

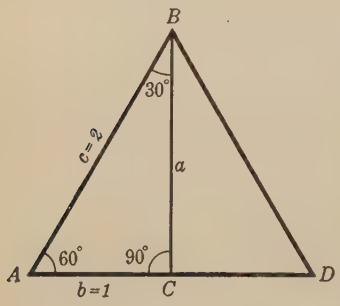

FIG. 21

$$\sin 60° = \frac{a}{c} = \tfrac{1}{2}\sqrt{3},$$

$$\cos 60° = \frac{b}{c} = \tfrac{1}{2},$$

$$\tan 60° = \frac{a}{b} = \sqrt{3},$$

$$\cot 60° = \frac{b}{a} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3},$$

$$\sec 60° = \frac{c}{b} = 2,$$

$$\csc 60° = \frac{c}{a} = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}.$$

Functions of 30° are equal to the corresponding cofunctions of 60°, which is the complementary angle. Therefore

$$\sin 30° = \tfrac{1}{2}, \qquad \cos 30° = \tfrac{1}{2}\sqrt{3},$$
$$\tan 30° = \tfrac{1}{3}\sqrt{3}, \qquad \cot 30° = \sqrt{3},$$
$$\sec 30° = \tfrac{2}{3}\sqrt{3}, \qquad \csc 30° = 2.$$

The student should memorize the following table:

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
$\cos\theta$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$

15. Tables of trigonometric functions. — *Four-place tables.* In Table IV, pages 92–96 of the tables, are columns headed "Sine," "Tangent," "Cotangent," "Cosine." These give the values of these four functions to four decimal places for acute angles at intervals of

10′. The secant and cosecant, if needed, can be found as reciprocals of the cosine and sine.

There are two problems involved in the use of the tables: (*a*) given an angle, to find the corresponding value of a function, and (*b*) given the value of a function, to find the angle to which it corresponds. The methods are illustrated in the following examples.

16. A function of a given angle. —

EXAMPLE 1. Find sin 28° 20′. On page 95, in the column headed "Degrees," we find the entry 28° 20′ and opposite this entry, in the column headed "Sine," we find the number .4746. So sin 28° 20′ = .4746. In the same line, in the other columns, we find tan 28° 20′ = .5392, cot 28° 20′ = 1.8546, cos 28° 20′ = .8802.

EXAMPLE 2. Find sin 56° 40′. In the column marked "Degrees" at the top the entries go only to 45°. But in the column at the right, marked "Degrees" at the bottom, the entries read *upward* from 45° to 90°. In fact, for any row in the table, the angle at the left and the angle at the right are complementary angles, and consequently an entry in any column gives one function of the angle marked at the left and the corresponding cofunction of the complementary angle which is marked at the right. By this arrangement each entry of the table is made to do double service, with consequent economy in printing and using the table. In seeking a function of an angle entered in the column at the right, we must use the designation of the function which is printed at the bottom. The entry for 56° 40′ is on page 95, and we find sin 56° 40′ = .8355. Likewise tan 56° 40′ = 1.5204, cot 56° 40′ = .6577, cos 56° 40′ = .5495.

EXERCISES

From Table IV find the sine, cosine, tangent and cotangent of each of the following angles.

1. sin 11° 20′.	**2.** cos 6° 50′.	**3.** tan 17° 0′.
4. cot 23° 40′.	**5.** sin 33° 30′.	**6.** cos 40° 10′.
7. sin 81° 40′.	**8.** cos 69° 0′.	**9.** tan 49° 50′.
10. cot 76° 50′.	**11.** tan 61° 0′.	**12.** cot 52° 20′.

17. Interpolation. — When we wish to find a number which is not given in the tables, but lies between two entries of a table, we resort to the method of *interpolation.*

EXAMPLE 1. Find sin 34° 27′. The angle 34° 27′ lies between the two entries 34° 20′ and 34° 30′. The difference between two successive entries in the table is called the *tabular difference.* From the tables we find

$$\begin{aligned} \sin 34° 20' &= .5640 \\ \sin 34° 30' &= .5664 \\ \hline \text{Tabular Difference} &= .0024. \end{aligned}$$

It is not correct to say that the sine of an angle is proportional to the angle. But for small changes in the angle, the change in the sine is nearly proportional to the change in the angle. And we may assume that *since* 34° 27′ *is "seven-tenths of the way" from* 34° 20′ *to* 34° 30′ *then* sin 34° 27′ *will also be "seven-tenths of the way" from* .5640 *to* .5664. We must therefore add to .5640 seven-tenths of the tabular difference .0024. That is

$$\sin 34° 27' = .5640 + .7 \times .0024 = .5640 + .00168 = .56568.$$

We must remember, however, that the tables themselves are accurate only to four decimal places. When we say that sin 34° 20′ = .5640 we mean that sin 34° 30′ is nearer to this number than it is either to .5639 or to .5641. In other words, the entries in the table have been "rounded off" to four places. We should be wrong to expect greater accuracy in our interpolated result than exists in the tables themselves. We therefore "round off" our number to four places, and write sin 34° 27′ = .5657. The work of interpolation should be done mentally about as follows: "Sin 34° 20′ = .5640; the tabular difference is 24 (in the last places); .7 of 24 is 16.8, which we call 17; this number added to .5640 gives sin 34° 27′ = .5657." Only the result is to be written down.

EXAMPLE 2. Find cos 34° 27′. The process is the same as in the preceding example, except that since the cosine *decreases* when the angle increases, we must now *subtract* seven-tenths of the tabular difference from cos 34° 20′. We find cos 34° 20′ = .8258; the tabular difference is (8258 − 8241) = 17; .7 × 17 = 11.9, which we call 12 and subtract from 8258. The result is cos 34° 27′ = .8246. The student should verify the following results: tan 34° 27′ = .6860, cot 34° 27′ = 1.4577.

EXAMPLE 3. Find cot 64° 45′. From the table cot 64° 40′ = .4734. The tabular difference is 35, and five-tenths of the difference is 17.5. This number is equally near to 17 and to 18, and there is no logical reason for choosing one in preference to the other. It is well, however, to have some fixed practice, and it is the custom among computers in interpolating to *round off .5 so that the result written down is the even rather than the possible odd number. We shall adhere to this rule throughout the book.* In a long piece of computation, the cases in which the result is increased are likely to be nearly balanced by the cases in which the result is diminished. In the present case, if we round off 17.5 to 18, the result is cot 64° 45′ = .4716, which, being even, is accepted as the result. It is the result, rather than the added difference (17 or 18), which is to be even, and the rule is applicable only in case of logical doubt, when it is .5 which is to be rounded off.

EXERCISES

From Table IV find the values of the following functions.

1.	$\cos 8° 13'$.	*Ans.*	.9898.	**2.**	$\tan 19° 28'$.
3.	$\cot 24° 14'$.	*Ans.*	2.2217.	**4.**	$\sin 37° 31'$.
5.	$\tan 41° 56'$.	*Ans.*	.8983.	**6.**	$\cot 10° 9'$.
7.	$\sin 48° 16'$.	*Ans.*	.7462.	**8.**	$\cos 57° 57'$.
9.	$\sin 74° 31'$.	*Ans.*	.9637.	**10.**	$\cos 64° 34'$.
11.	$\tan 45° 19'$.	*Ans.*	1.0111.	**12.**	$\cot 60° 55'$.
13.	$\sin 9° 35'$.	*Ans.*	.1664.	**14.**	$\tan 27° 45'$.

18. The angle corresponding to a given function. —

EXAMPLE 1. Given $\sin \theta = .5225$, find the value of θ. Looking in the column headed "Sine," we find the given entry on page 95, and write $\theta = 31° 30'$.

EXAMPLE 2. Find θ if $\tan \theta = 3.9136$. This entry is found on page 93 in the column marked "Tangent" at the bottom. It corresponds to $\theta = 75° 40'$.

EXAMPLE 3. Find θ if $\sin \theta = .5531$. This entry does not occur in the tables, but is between $.5519 = \sin 33° 30'$ and $.5544 = \sin 33° 40'$. We can arrange the numbers as follows.

Angle	*Sine*	*Difference*
$33° 30'$	.5519	12
θ	.5531	
$33° 40'$	.5544	25

Therefore $\sin \theta$ is "twelve twenty-fifths of the way" from $\sin 33° 30'$ to $\sin 33° 40'$, and we conclude that θ is "twelve twenty-fifths of the way" from $33° 30'$ to $33° 40'$. Using decimals we get $12/25 = .48$ which we call .5. To $33° 30'$ we therefore add $.5 \times 10'$, and get $\theta = 33° 35'$.

EXAMPLE 4. Find θ, given $\cos \theta = .5164$. This number appears between the entries for $58° 50'$ and $59° 0'$. The tabular difference is 25; our difference is $75 - 64 = 11$. The proportional part is $11/25$, which we call .4, and the angle is $\theta = 58° 54'$.

EXERCISES

From Table IV find an angle corresponding to each of the following functions, giving its value correctly to the nearest minute: (Answers are given after Example **28**.)

1. $\sin\theta = .0669$.	**2.** $\sin\theta = .6338$.	**3.** $\cos\theta = .9781$.
4. $\cos\theta = .9276$.	**5.** $\tan\theta = .1974$.	**6.** $\tan\theta = .6128$.
7. $\cot\theta = 1.3190$.	**8.** $\sin\theta = .9001$.	**9.** $\cos\theta = .5854$.
10. $\cos\theta = .2476$.	**11.** $\tan\theta = 1.6003$.	**12.** $\cot\theta = .3057$.
13. $\sin\theta = .5336$.	**14.** $\sin\theta = .1341$.	**15.** $\cos\theta = .9888$.
16. $\cos\theta = .8456$.	**17.** $\tan\theta = .5727$.	**18.** $\tan\theta = .8251$.
19. $\cot\theta = 1.5927$.	**20.** $\cot\theta = 2.8568$.	**21.** $\sin\theta = .8015$.
22. $\sin\theta = .8776$.	**23.** $\cos\theta = .6521$.	**24.** $\cos\theta = .0939$.
25. $\tan\theta = 1.5007$.	**26.** $\tan\theta = 11.943$.	**27.** $\cot\theta = .2573$.

28. Find the values of the other tabulated functions for each of the angles of Examples **1** to **27**.

Ans.	θ	$\sin\theta$	$\tan\theta$	$\cot\theta$	$\cos\theta$
1.	3° 50′	——	.0670	14.924	.9978.
3.	12° 0′	.2079	.2126	4.7046	——
13.	32° 15′	——	.6310	1.5849	.8458.
15.	8° 35′	.1492	.1510	6.6259	——
17.	29° 48′	.4970	——	1.7461	.8678.
19.	32° 7′	.5316	.6277	——	.8470.
21.	53° 16′	——	1.3400	.7463	.5981.
23.	49° 18′	.7581	1.1626	.8601	——
25.	56° 19′	.8321	——	.6665	.5546.
27.	75° 34′	.9684	3.8855	——	.2493.

19. A five-place table of trigonometric functions. — Table III is similar to Table IV, except that angles are given for every minute and the corresponding values of the functions are given to five places of decimals. It is proper, with this table, to interpolate to or from an angle expressed correctly to the nearest tenth of a minute.

EXAMPLE 1. Find sin 39° 41.3′. On page 87 of the tables, we find

sin 39° 41′	= .63854
sin 39° 42′	= .63877
Tabular difference	= 23

Then .3 × 23 = 6.9 which we call 7. We add this to sin 39° 41′, and so obtain sin 39° 41.3′ = .63861.

EXAMPLE 2. Find tan 62° 32.8′. The nearest entry is on page 81. The tabular difference is 14, .8 of which is 11.2. We find tan 62° 32.8′ = 1.9248.

EXAMPLE 3. Find θ, given cot θ = .61831. From page 83,

cot 58° 16′	= .61842	cot 58° 16′	= .61842
cot 58° 17′	= .61801	cot θ	= .61831
Tabular difference	= 41.	Our difference	= 11.

Proportional part = 11/41 or approximately .3. Therefore θ = 58° 16.3′.

EXAMPLE 4. Find θ, given cos θ = .96240. From page 75, the tabular difference = 8, our difference = 6. The proportional part is 6/8, which we round off to .8. Therefore θ = 15° 45.8′.

EXERCISES

Find from Table III the sine, cosine, tangent and cotangent of each of the following angles.

1. 14° 43′.	**2.** 7° 11′.	**3.** 51° 46′.
4. 69° 18′.	**5.** 44° 26′.	**6.** 45° 34′.
7. 31° 58.4′.	**8.** 11° 27.8′.	**9.** 58° 17.3′.
10. 74° 31.9′.	**11.** 77° 23.2′	**12.** 23° 0.6′.
13. 41° 0.2′.	**14.** 81° 40.3′.	**15.** 51° 16.6′.
16. 48° 18.8′.	**17.** 35° 55.5′.	**18.** 28° 8.5′.

Ans.	sin θ	tan θ	cot θ	cos θ
1.	.25404	.26266	3.8073	.96719.
3.	.78550	1.2693	.78786	.61887.
7.	.52953	.62422	1.6020	.84830.
9.	.85070	1.6184	.61789	.52564.
17.	.58672	.72454	1.3802	.80978.

In each of the following cases find θ (between 0° and 90°) correct to the nearest tenth of a minute, and find the values of the other tabulated functions.

19. $\sin\theta = .67666$.	**20.** $\sin\theta = .54000$.	**21.** $\cos\theta = .74002$.
22. $\cos\theta = .95337$.	**23.** $\tan\theta = .18895$.	**24.** $\tan\theta = .93906$.
25. $\cot\theta = 1.2145$.	**26.** $\cot\theta = 2.3201$.	**27.** $\sin\theta = .81412$.
28. $\sin\theta = .92287$.	**29.** $\cos\theta = .70360$.	**30.** $\cos\theta = .59342$.
31. $\tan\theta = 1.4559$.	**32.** $\tan\theta = 1.2002$.	**33.** $\cot\theta = .78269$.
34. $\sin\theta = .48881$.	**35.** $\sin\theta = .46521$.	**36.** $\sin\theta = .62083$.
37. $\sin\theta = .88791$.	**38.** $\sin\theta = .79629$.	**39.** $\cos\theta = .81356$.
40. $\cos\theta = .97314$.	**41.** $\cos\theta = .64137$.	**42.** $\cos\theta = .25286$.
43. $\tan\theta = .64137$.	**44.** $\tan\theta = .27933$.	**45.** $\tan\theta = 1.3860$.
46. $\tan\theta = 2.7844$.	**47.** $\tan\theta = 1.0008$.	**48.** $\cot\theta = 1.8249$.
49. $\cot\theta = 1.2398$.	**50.** $\cot\theta = .97452$.	**51.** $\cot\theta = .06689$.

Ans. **19.** 42° 35′. **21.** 42° 16′. **23.** 10° 42′.
33. 51° 57′. **35.** 27° 43.4′. **39.** 35° 33.3′.
43. 32° 40.5′. **47.** 45° 1.3′. **49.** 38° 53.4′.

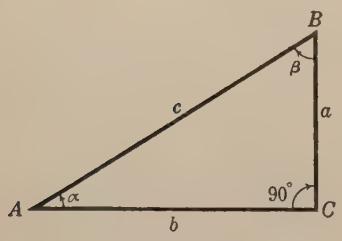

FIG. 22

20. The solution of a right triangle. — Besides its right angle, a right triangle possesses five *parts*, which are its two acute angles and its three sides. If two of these parts are known, at least one of them being a side, it is possible to determine the other three parts. The determination of the unknown parts constitutes a *solution* of the triangle.

For the solution of the right triangle ABC, we have the following formulas.

(1) $a^2 + b^2 = c^2$, (2) $\alpha + \beta = 90°$,

(3) $\sin\alpha = \dfrac{a}{c} = \cos\beta$, (4) $\cos\alpha = \dfrac{b}{c} = \sin\beta$,

(5) $\tan \alpha = \frac{a}{b} = \cot \beta$ (6) $\cot \alpha = \frac{b}{a} = \tan \beta.$

In order to determine an unknown part, select and use one of the formulas which involves that part and no other unknown. As a check against gross errors draw the figure approximately to scale and compare the results with the figure. Methods of checking more closely will be explained in connection with the solutions of triangles by logarithms.

EXAMPLE 1. Solve the right triangle ABC, given

$$c = 21.33, \quad \alpha = 32° 20'.$$

From formula (2), $\beta = 90° - 32° 20' = 57° 40'.$

From (3), $a = c \times \sin \alpha = 21.33 \times \sin 32° 20' = 21.33 \times .5348 = 11.41.$

From (4), $b = c \times \cos \alpha = 21.33 \times .8450 = 18.02.$

EXAMPLE 2. Solve the right triangle ABC, given

$$a = 10.2, \quad b = 18.3.$$

From formula (5), $\tan \alpha = \frac{a}{b} = \frac{10.2}{18.3} = .5574.$

Therefore, by Table IV, $\alpha = 29° 8', \beta = 90° - 29° 8' = 60° 52'.$

By formula (3), $c = \frac{a}{\sin \alpha} = \frac{10.2}{\sin 29° 8'} = \frac{10.2}{.4869} = 20.9.$

EXAMPLE 3. — A radio aërial is to be stretched from the corner of a house to the top of a pole 62 feet away. By sighting, it is observed that the wire will make an angle of 15° with the horizontal. How long must the wire be, if 8% be added to allow for sag and contraction?

In Figure 23, $b = 62$ feet, $\alpha = 15°$.

$$c = \frac{b}{\cos 15°} = \frac{62}{.9659} = 64.2 \text{ feet.}$$

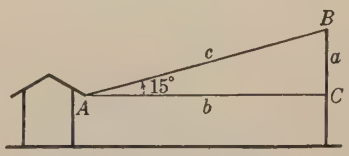

FIG. 23

Adding 8% or 5.1 feet to this, we get 69.3 feet as the length of the wire.

EXERCISES

In the following exercises give angles to the nearest minute. Solve the following right triangles, given:

1. $c = 16.4$, $\alpha = 24° 50'$.

   Ans. $a = 6.89$, $b = 14.88$, $\beta = 65° 10'$.

2. $c = 258$, $\alpha = 57° 30'$.

3. $c = 423.6$, $\alpha = 49° 27'$.

   Ans. $a = 321.85$, $b = 275.38$, $\beta = 40° 33'$.

4. $c = 60.24$, $\alpha = 28° 43'$.

5. $a = 24.6$, $b = 40.4$.

   Ans. $\alpha = 31° 20'$, $\beta = 58° 40'$, $c = 47.3$.

6. $a = 9.07$, $b = 5.63$.

7. $a = 6011$, $b = 8203$.

   Ans. $\alpha = 36° 14'$, $\beta = 53° 46'$, $c = 10170$.

8. $a = .8104$, $b = .5302$.

9. $a = 3.81$, $c = 5.07$.

   Ans. $\alpha = 48° 43'$, $\beta = 41° 17'$, $b = 3.35$.

10. $a = 511.2$, $c = 810.3$.

11. $b = .04318$, $c = .07004$.

    Ans. $\alpha = 51° 57'$, $\beta = 38° 3'$, $a = .05516$.

12. $b = 62.31$, $c = 100.44$.

13. $a = 25.3$, $\alpha = 38° 40'$.

    Ans. $\beta = 51° 20'$, $c = 40.49$, $b = 31.62$.

14. $a = .707$, $\alpha = 65° 22'$.

15. $a = 324.4$, $\beta = 21° 16'$.

    Ans. $\alpha = 68° 44'$, $c = 348.1$, $b = 126.3$.

16. $a = 4.021$, $\beta = 49° 9'$.

21. Angles of elevation and depression. — Suppose that an observer is at a point O, and a certain object is at a point P. The line OP is then called the *line of sight* from the observer to the object. Let OH be drawn horizontally in the vertical plane through OP.

Then the angle HOP which the line of sight makes with the horizontal is called the *angle of elevation* of the object P from O if P is higher than O, and it is called the *angle of depression* of P from O, if P is lower than O.

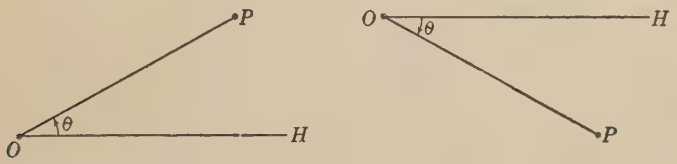

FIG. 24

EXERCISES

These exercises may be worked either with or without logarithms.

1. From a point on the ground 200 feet from the foot of a flagstaff, the angle of elevation of the top is 30°. How high is the flagstaff?

Ans. $115\frac{1}{2}$ ft.

2. From a point 1150 feet from the foot of a cliff, the angle of elevation of the top is 20°. How high is the cliff?

3. If the angle of elevation of the sun is 30° how long a shadow will be cast on the ground by a man 6 feet tall? *Ans.* 10.4 ft.

4. If the angle of elevation of the sun is 60° how long a shadow will be cast on the ground by a spire 140 feet high?

5. Find the angle of elevation of the sun if an object 20 feet high casts a shadow 60 feet long. *Ans.* 18° 26′.

6. What is the angle of elevation of the sun if an object 40 feet high casts a shadow 60 feet long?

7. When the angle of elevation of the sun is 15° 1′, the shadow of the Great Pyramid reaches to a point about 1790 feet from the center of the square base. How high is the pyramid?

Ans. About 480 ft.

Historical note: One of the earliest uses of trigonometry was in determining the heights of the pyramids and other objects by measuring their shadows.

8. The angle of elevation of the sun is 60° and the shadow of a vertical post is 8.7 feet long. How high is the post?

9. The height of the Great Pyramid is about 480 feet and a side of the square base is about 760 feet. What are the angles of inclination of a face and of an edge? *Ans.* 51° 38′, 41° 46′

10. How high is a kite when the length of the string is 600 feet and its angle of elevation is 32°?

11. A ladder 36 feet long reaches from the ground to a window 33 feet high. How far from the wall is its foot, and what angle does it make with the horizontal? *Ans.* 14.4 ft., 66° 27′.

12. A guy wire is stretched from the top of a telephone pole 39 feet high to a stake on the ground 23 feet from the pole. Find the length of this wire and the angle which it makes with the pole.

13. From a lighthouse which stands 109.7 feet above the level of a lake, the angle of depression of a small object floating on the surface of the water is 19° 23′. Find the distance of the object from the base of the lighthouse. *Ans.* 311.8 ft.

14. A surveyor's instrument stands 4.7 feet above the ground and 187.3 feet from the foot of a vertical flagstaff. From the instrument the angle of elevation of the top of the flagstaff is 16° 42′. How high is the flagstaff?

15. A radio aërial is stretched from a point on a house 32.8 feet high to the top of a pole 51.4 feet high. The horizontal distance between the points is 63.7 feet. Find the angle which the aërial makes with the horizontal, and its length, if 0.4 foot be added to allow for contraction. *Ans.* 16° 17′, 66.7 ft.

16. A flagstaff stands on top of a tower. From a point on the level ground 221 feet from the base of the tower, the angle of elevation of the top of the flagstaff is 19° 15′, and the angle of elevation of the bottom of the flagstaff is 15° 40′. Find the height of the flagstaff.

17. From the window of an office building the angle of elevation of the top of a church steeple is 21° 35′ and the angle of depression of a point on the ground directly below the steeple is 32° 20′. The

horizontal distance between window and church is 123 feet. How high is the steeple, and how far above the ground is the window?

Ans. 127 ft., 78 ft.

18. From the top of a lighthouse which stands 217 feet above the level of the sea, the angle of depression of the top of the mast of an anchored vessel is 23° 10′ and the angle of depression of a point on the vessel's waterline is 34° 35′. Find the height of the vessel and its distance from the base of the lighthouse.

19. A tower rises from the bank of a river. A man standing directly opposite on the other bank observes the angle of elevation of the top of the tower to be 58° 45′. He then goes back 102 feet from the river bank, and finds the angle of elevation of the top of the tower to be 26° 25′. Find the height of the tower and the width of the stream. *Ans.* 72.5 ft., 44 ft.

20. From a point on a horizontal plain the angle of elevation of a cross on top of a mountain is 21° 43′. After walking 1272 feet directly toward the mountain the observer finds the angle of elevation to be 26° 27′. How high is the mountain above the plain?

21. Two buoys are directly in line with a cliff which stands 282 feet above the water. Their angles of depression from the top of the cliff are 42° 25′ and 37° 15′. How far apart are the buoys and what are their horizontal distances from the cliff?

Ans. 62 ft., 371 ft., 309 ft.

22. Two buoys are anchored in line with a cliff and are 325 feet apart. At high tide the angles of depression from the top of the cliff are 38° 45′ and 27° 10′ respectively. At low tide the angles are 40° 20′ and 28° 30′. How high is the tide?

23. In order to find the distance from a boat dock, A, to an island, B, a line, AC, 1000 feet long, was measured along the shore in a direction at right angles to AB. The angle ACB was measured and found to be 63° 25′. What was the required distance? *Ans.* 1998 ft.

24. In Ex. **23** if there is a lighthouse 180 feet high at C, what is the angle ADB measured by an observer D at the top of the lighthouse?

25. A railroad track rises 1 foot for every 25 feet measured along

the track. What angle does the track make with the horizontal? How many feet does it rise in a mile measured along the track? If it costs the road 1 cent to raise 330 tons of freight 1 foot (more than it does to move it on the level), what will this grade cost the road in moving 100,000 tons of freight 1 mile?

Ans. 2° 18′, 211.2 ft., $640.

26. In a certain department store there is an escalator or moving stairway which moves at the rate of 250 feet per minute and is inclined at 40° to the horizontal. How long will it take to carry a shopper from one floor to the next, the height of one story being 19 feet?

27. The minute hand of a certain clock is 4.5 inches long. At 10 minutes past 3 o'clock, the line joining the ends of the two hands is perpendicular to the hour hand. What is the length of the hour hand? *Ans.* 3.7 in.

28. Work Ex. **27**, taking the length of the minute hand as 5.4 inches and the time as 4 minutes before 10 o'clock.

29. At what time between 2 o'clock and 10 minutes past 2 will the line joining the ends of the hands of a clock be perpendicular to the hour hand, if the minute hand is 4.8 inches long and the hour hand is 3.1 inches long? *Ans.* 1.9 mins. past 2.

30. At what time between 11 and 11:10 will the hands be in the relative positions indicated in Ex. **29**, if the minute hand is 6.3 inches long, and the hour hand is 4.3 inches long?

31. In each step of a certain stairway the tread (or horizontal part) is 9 inches and the riser (or vertical part) is 8 inches. What angle does the banister make with the horizontal? *Ans.* 41° 38′.

32. If the tread of a stairway is 11 inches and if the banister makes an angle of 39° 27′, how high is the riser?

33. The radius of a circle is 20 inches. Find the length of a side of an inscribed regular pentagon (5-sided polygon). *Ans.* 23.51 in.

34. The radius of a circle is 20 inches. Find the perimeter of an inscribed regular octagon.

35. A side of an equilateral triangle is 20 inches. Find the radius of the inscribed circle. *Ans.* $r = 10/\sqrt{3} = 5.77$ in.

36. The area of an equilateral triangle is 300 square inches. Find the area of the inscribed circle.

37. Two boats are anchored 2500 feet apart. An airplane flies across the line joining them. Just as it does so its angle of elevation from one boat is 50° and its angle of elevation from the other is 60°. Find the height at which it was flying. *Ans.* 1765 ft.

38. The angle of elevation of an approaching airplane from a landing field is 22°. The plane glides straight toward the landing field and arrives 90 seconds after it is first sighted. If its average speed during that time is 40 miles per hour, how high was it when first sighted?

39. A boat is anchored near a bridge. From a point on its deck 15 feet above the water, the angle of elevation of the top of the bridge is 39°, while the angle of depression of its image reflected in the water is 54°. Find the height of the bridge above the water and the distance of the boat from the bridge. *Ans.* Height 58 ft., dist. 53 ft.

40. An airplane is flying horizontally directly away from a station on the ground at the rate of 80 miles an hour. At a certain instant its angle of elevation from the station is 40°, and 30 seconds later its angle of elevation is 25°. Find the height at which it is flying.

CHAPTER III

LOGARITHMS

22. The uses of logarithms. — In solving triangles, and in many other problems, the work of numerical computation can be greatly lessened by the use of logarithms. We shall see that logarithms enable us to replace the processes of multiplication and division by the simpler ones of addition and subtraction, and to replace the taking of powers and roots by multiplication and division. *Logarithms*, we shall find, *are exponents*. For that reason in the next article we recall the fundamental laws of exponents.

23. Laws of exponents. — Let a be any positive number and x and y any two numbers. Then

I. $a^x \cdot a^y = a^{x+y}$.

EXAMPLE. $3^5 \cdot 3^2 = 3^7$.

II. $\dfrac{a^x}{a^y} = a^{x-y}$.

EXAMPLE. $3^5/3^2 = 3^3$.

III. $(a^x)^y = a^{xy}$.

EXAMPLE. $(3^5)^2 = 3^{10}$.

IV. $\sqrt[y]{a^x} = a^{\frac{x}{y}}$.

EXAMPLE. $\sqrt{3^5} = 3^{\frac{5}{2}}$.

V. $a^0 = 1$.

EXAMPLE. $3^0 = 1$.

VI. $a^{-x} = \dfrac{1}{a^x}$, and $\dfrac{1}{a^x} = a^{-x}$.

EXAMPLE. $\dfrac{1}{81} = \dfrac{1}{3^4} = 3^{-4}$.

VII. If a is any number greater than 1 and if y is positive, it is always possible to find a real exponent x such that

$$a^x = y.$$

But if y is negative no real value of x exists satisfying this relation.

24. Definition of logarithms. — If a, x, and y are three numbers such that

(1) $$a^x = y,$$

the exponent x is said to be the logarithm of y to the base a, which we write

(2) $$x = \log_a y.$$

The equations (1) and (2) are different ways of expressing the same relation among the three numbers a (the *base*), y (the *number*) and x (the *exponent* or *logarithm*). Equation (1) is in *exponential* form; while (2) says the same thing in *logarithmic* form.

Definition: The logarithm of a number to a given base is the exponent to which the base must be raised to yield the number. By VII, Article **23**, if a is greater than 1, any positive number has a logarithm to the base a, but a negative number has no real logarithm.

EXAMPLES. 1. Since $2^3 = 8$, $3 = \log_2 8$.

2. Since $3^{-4} = \frac{1}{81}$, $-4 = \log_3 (\frac{1}{81})$.

3. Since $\sqrt[3]{4} = 2^{\frac{2}{3}}$, $\frac{2}{3} = \log_2 (\sqrt[3]{4})$.

4. Since $a^1 = a$, $\log_a a = 1$.

5. Since $a^0 = 1$, $\log_a 1 = 0$.

ORAL EXERCISES

Change the following statements from exponential to logarithmic form.

1. $2^5 = 32$. *Ans.* $\log_2 32 = 5$. **2.** $3^2 = 9$.

3. $3^4 = 81$. *Ans.* $\log_3 81 = 4$. **4.** $5^3 = 125$.

5. $10^2 = 100$. **6.** $10^4 = 10{,}000$. **7.** $6^3 = 216$.

8. $\sqrt{4} = 2$. *Ans.* $\log_4 2 = \frac{1}{2}$. **9.** $\sqrt{9} = 3$.

10. $\sqrt[3]{27} = 3$. **11.** $\sqrt[5]{32} = 2$. **12.** $\sqrt{100} = 10$.

13. $2^{-2} = \frac{1}{4}$. *Ans.* $\log_2 (\frac{1}{4}) = -2$. **14.** $3^{-1} = \frac{1}{3}$.
15. $10^{-2} = .01$. **16.** $10^{-1} = 0.1$. **17.** $10^{-3} = .001$.
18. $3^0 = 1$. **19.** $7^0 = 1$. **20.** $2^1 = 2$.

In each of the following examples, find the value of x.

21. $x = \log_2 16$. *Ans.* $x = 4$. **22.** $x = \log_6 36$.
23. $x = \log_{10} 100$. **24.** $x = \log_{10} .01$. **25.** $x = \log_9 81$.
26. $x = \log_4 8$.

Ans. We have $4^x = 8$, or $2^{2x} = 2^3$. $\therefore\ 2x = 3,\ x = \frac{3}{2}$.

27. $x = \log_{27} 81$. **28.** $x = \log_{16} 2$. **29.** $x = \log_{36} 216$.
30. $x = \log_5 \left(\frac{1}{\sqrt{125}}\right)$. *Ans.* $x = -\frac{3}{2}$. **31.** $x = \log_2 \frac{1}{\sqrt[3]{32}}$.
32. $\log_x 8 = 3$. *Ans.* $x = 2$. **33.** $\log_x 25 = 2$.
34. $\log_x 16 = \frac{4}{5}$.

Ans. $x^{\frac{4}{5}} = 16$. $\therefore\ x^{\frac{1}{5}} = \sqrt[4]{16} = 2,\ x = 2^5 = 32$.

35. $\log_x 81 = \frac{4}{3}$. **36.** $\log_x (\frac{27}{8}) = 3$. **37.** $\log_x \frac{1}{64} = -\frac{3}{2}$.
38. $\log_x .001 = -\frac{3}{2}$. **39.** $\log_x 256 = \frac{4}{3}$. **40.** $\log_x 25 = \frac{2}{3}$.
41. $\log_3 x = -5$. *Ans.* $x = \frac{1}{243}$. **42.** $\log_5 x = 3$.
43. $\log_{25} x = -\frac{3}{2}$. **44.** $\log_{10} x = 2$. **45.** $\log_4 x = \frac{5}{2}$.
46. $\log_a x = 0$. **47.** $\log_a x = 1$. **48.** $\log_a x = b$.

Show that

49. $a^{\log_a x} = x$. **50.** $\log_a (a^x) = x$. **51.** $\log_a \left(\frac{1}{a}\right) = -1$.

25. The fundamental theorems of logarithms. — Corresponding to the four laws of exponents we have the following four theorems.

I. *Multiplication. — The logarithm of the product of two numbers is equal to the sum of the logarithms of the two numbers:*

$$\log_a (M \cdot N) = \log_a M + \log_a N.$$

Proof. Let M and N be any two positive numbers, and let

$$x = \log_a M \qquad \text{and} \qquad y = \log_a N.$$

Then, from the definition of logarithms,

$$a^x = M \qquad \text{and} \qquad a^y = N.$$

Therefore, from I, Article 23,

$$M \cdot N = a^x \cdot a^y = a^{x+y}.$$

Consequently

$$\log_a (M \cdot N) = x + y = \log_a M + \log_a N.$$

It is easily seen that the logarithm of the product of *any* number of factors is equal to the sum of their logarithms. This theorem enables us to replace the laborious process of multiplying two or more numbers, by the comparatively easy one of adding their logarithms.

II. *Division. — The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor:*

$$\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N.$$

Proof. Using the same notation as before, we have

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y},$$

by Law II for exponents. This is equivalent to saying that

$$\log_a \left(\frac{M}{N}\right) = x - y = \log_a M - \log_a N,$$

as we wished to show.

This theorem permits us to substitute the simple process of subtraction for the more arduous one of division. By combining this theorem with the preceding one, we see that *the logarithm of any fraction is equal to the sum of the logarithms of the factors of the numerator minus the sum of the logarithms of the factors of the denominator.*

Example.

$$\log_a \left(\frac{32 \times 12.7 \times 194}{17 \times 542}\right) = \log_a 32 + \log_a 12.7 + \log_a 194 - \log_a 17 - \log_a 542.$$

III. *Involution. — The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power:*

$$\log_a (M^n) = n \log_a M.$$

Proof. Let

$$x = \log_a M,$$

so that

$$a^x = M.$$

Then

$$M^n = (a^x)^n = a^{nx},$$

by Law III for exponents. Therefore, by the definition of the logarithm of a number,

$$\log_a (M^n) = nx = n \log_a M,$$

as we were to prove.

EXAMPLE. To find 17^{19}, we notice that $\log_{10} (17^{19}) = 19 \log_{10} 17$ which, as we shall see, is easily obtained.

IV. *Evolution. — The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root:*

$$\log_a (\sqrt[n]{M}) = \frac{\log_a M}{n}.$$

Proof. Since, by Law IV of exponents,

$$\sqrt[n]{M} = M^{\frac{1}{n}},$$

by the preceding theorem we have

$$\log_a (\sqrt[n]{M}) = \log_a (M^{\frac{1}{n}}) = \frac{1}{n} \log_a M,$$

as we wished to show.

EXAMPLE. To find $\sqrt[7]{247}$ directly is a difficult task, but we shall see that to find $\sqrt[7]{247}$ by logarithms is a simple matter, since

$$\log_{10} \sqrt[7]{247} = \tfrac{1}{7} \log_{10} 247.$$

EXERCISES

Given $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$, find the following logarithms.

1.	$\log_{10} 6$.	*Ans.* 0.778.	**2.**	$\log_{10} \frac{3}{2}$.
3.	$\log_{10} \frac{2}{3}$.	*Ans.* -0.176.	**4.**	$\log_{10} 20$.
5.	$\log_{10} 10$.	*Ans.* 1.000.	**6.**	$\log_{10} 1$.
7.	$\log_{10} 30$.	*Ans.* 1.477.	**8.**	$\log_{10} 60$.
9.	$\log_{10} \frac{1}{2}$.	*Ans.* -0.301.	**10.**	$\log_{10} \frac{1}{3}$.
11.	$\log_{10} 2^2$.	*Ans.* 0.602.	**12.**	$\log_{10} 3^2$.
13.	$\log_{10} 9$.	*Ans.* 0.954.	**14.**	$\log_{10} 4$.
15.	$\log_{10} 12$.	*Ans.* 1.079.	**16.**	$\log_{10} 18$.
17.	$\log_{10} 8$.	*Ans.* 0.903.	**18.**	$\log_{10} 27$.
19.	$\log_{10} 40$.	*Ans.* 1.602.	**20.**	$\log_{10} 180$.
21.	$\log_{10} 5$.	*Ans.* 0.699.	**22.**	$\log_{10} \frac{1}{4}$.
23.	$\log_{10} \frac{1}{9}$.	*Ans.* -0.954.	**24.**	$\log_{10} 36$.
25.	$\log_{10} 48$.	*Ans.* 1.681.	**26.**	$\log_{10} 72$.
27.	$\log_{10} 45$.	*Ans.* 1.653.	**28.**	$\log_{10} 225$.
29.	$\log_{10} \sqrt[3]{2}$.	*Ans.* 0.100.	**30.**	$\log_{10} \sqrt{3}$.
31.	$\log_{10} \sqrt[5]{3}$.	*Ans.* 0.095.	**32.**	$\log_{10} \sqrt[4]{2}$.
33.	$\log_{10} \sqrt[17]{27}$.	*Ans.* 0.084.	**34.**	$\log_{10} \sqrt[6]{6}$.

26. Logarithms to the base 10. — In numerical computations the most useful base of logarithms is the number 10. Hereafter when we speak of the logarithm of a number without specifying the base, it is to be understood that the base of the logarithm is 10. And instead of writing $\log_{10} N$ we shall write merely $\log N$.

EXAMPLES.

$$\log 100 = 2, \quad \log .1 = -1,$$
$$\log 10 = 1, \quad \log .01 = -2.$$

27. Characteristic and mantissa. — Any number can be written as the sum of a whole number or integer and a fraction. These parts, of course, may be zero, or positive or negative. It is always

possible, however, to take the fractional part of the number positive or zero.

EXAMPLES.

$$\begin{aligned} 4.231 &= .231 + 4. \\ 0.462 &= .462 + 0. \\ -0.563 &= .437 - 1. \\ 5 &= 0 + 5. \\ -2.694 &= .306 - 3. \end{aligned}$$

For use in computation a logarithm is always written in this way as the sum of a whole (or "integral") number (positive, negative, or zero) and a fractional part (positive or zero). The integral part of a logarithm is called its *characteristic*, and its positive (or zero) fractional part is called its *mantissa*.

EXAMPLES.

Logarithm.	Characteristic.	Mantissa.
$\log 2 = 0.30103.$	0.	.30103.
$\log 20 = 1.30103.$	1.	.30103.
$\log .5 = -0.30103 = .69897 - 1.$	−1.	.69897.
$\log 100 = 2.00000.$	2.	.00000.
$\log .05 = -1.30103 = .69897 - 2.$	−2.	.69897.

28. Significant digits. — The numbers 704200 and 0.07042 are said to have the same *sequence of significant digits*, 7042. The significant digits of a number are the figures to the right of which and to the left of which are no figures other than zeros.

In the next two paragraphs we shall make use of the following table and of the fact that *the logarithm of an integral power of* 10 *is a whole number.*

$$\log 1 = \log 10^0 = 0,$$

$$\begin{aligned} \log 10 &= \log 10^1 = 1, & \log .1 &= \log 10^{-1} = -1, \\ \log 100 &= \log 10^2 = 2, & \log .01 &= \log 10^{-2} = -2, \\ \log 1000 &= \log 10^3 = 3, & \log .001 &= \log 10^{-3} = -3, \\ &\text{etc.} & &\text{etc.} \end{aligned}$$

29. The mantissa. — *Theorem. If two numbers have the same significant digits, their logarithms have the same mantissas.*

Proof. The larger of the two numbers may be obtained from the other by moving the decimal point to the right. This is equivalent to multiplying the number by an integral power of 10. By Theorem I, Article **25**, this has the effect of adding to the logarithm of the number, the logarithm of an integral power of 10, which, as we have seen, is a whole number. A change in the position of the decimal point in a number therefore changes the whole part of its logarithm or characteristic, but it does not change its fractional part or mantissa.

EXAMPLES. Given $\log 7.042 = 0.84770$. Then

$$\log 70.42 = \log 7.042 + \log 10 = 0.84770 + 1 = 1.84770.$$
$$\log 704.2 = \log 7.042 + \log 100 = 0.84770 + 2 = 2.84770.$$
$$\log 0.7042 = 7.042 - \log 10 = 0.84770 - 1.$$
$$\log 0.07042 = \log 7.042 - \log 100 = 0.84770 - 2.$$

ORAL EXERCISES

Given $\log 6.071 = 0.7833$ and $\log 2.4 = 0.3802$,

give the values of the following logarithms.

1. $\log 60.71$. **2.** $\log 0.6071$. **3.** $\log 607.1$. **4.** $\log 24$. **5.** $\log 240$.
6. $\log 24000$. **7.** $\log 0.06071$. **8.** $\log 0.0006071$. **9.** $\log 0.0024$.

30. The characteristic. — We shall suppose that any number that we use in computation contains a decimal point; if it is not actually present, its presence is understood.

If a number has just one digit to the left of the decimal point (e.g., the number 7.21), the characteristic of its logarithm is zero. For the number (unless it be the number 1 itself, in which case its logarithm is zero) lies between 1 and 10. Therefore its logarithm is between $\log 1 = 0$ and $\log 10 = 1$, and has the characteristic 0.

Any positive number may be obtained from a number having one digit to the left of the decimal point by moving the decimal point a suitable number of places to the right or to the left. Each time that the decimal point is moved one place to the right the number is multiplied by 10, and $\log 10 = 1$ is added to the characteristic. We can say, then, that if a number has two digits to the left of the

decimal point (e.g., 72.1), the characteristic is 1; if there are three digits, the characteristic is 2; and if there are k digits to the left of the decimal point the characteristic is $(k - 1)$.

On the other hand, to move the decimal point one place to the left divides the number by 10 and reduces the characteristic of its logarithm by 1. It follows that if the decimal point immediately precedes the first significant digit (as in the number .721), the characteristic of the logarithm is -1; if the first significant digit is in the second place to the right of the decimal point (as in .0721), the characteristic is -2, etc.

Rule. If a number is greater than 1 *and has* k *digits to the left of the decimal point, the characteristic of its logarithm is* $k - 1$.

If a number is positive but less than 1, *and if its first significant digit is in the* k*th place to the right of the decimal point, the characteristic of its logarithm is* $-k$.

Examples. The characteristic of log 273.4 is 2; that of log 2.734 is 0; that of log 27.34 is 1; that of log 273400 is 5; that of log .2734 is -1; that of log .002734 is -3.

ORAL EXERCISES

What are the characteristics of the logarithms of the following numbers?

1. 5.76.	**2.** 609.1.	**3.** 2100.	**4.** 23600.
5. .637.	**6.** .78932.	**7.** .042.	**8.** .000632.

31. Notation. — Negative characteristics may be written in various ways. We may write

$$\log .08641 = .93656 - 2.$$

It would be incorrect to write this logarithm as -2.93656, for this would indicate that the mantissa is negative. Inasmuch, however, as $-2 = 8 - 10$, we may, *if we choose*, write

$$\log .08641 = 8.93656 - 10.$$

This method of writing negative characteristics in practice is the most convenient one, and will be used in this book. In a similar way we shall write

$$\log .8641 = 9.93656 - 10,$$
$$\log .0008641 = 6.93656 - 10, \text{ etc.}$$

EXERCISES

ORAL EXERCISES

Given $\log 3.078 = 0.48827$ and $\log 86.73 = 1.93817$, find the following logarithms.

1. $\log 307.8$. *Ans.* 2.48827. **2.** $\log 867.3$.
3. $\log 8.673$. **4.** $\log 3078$. **5.** $\log 30.78$.
6. $\log 307800$. *Ans.* 5.48827. **7.** $\log 30780$.
8. $\log 8673000$. **9.** $\log 86730$. **10.** $\log 8673$.
11. $\log 0.8673$. *Ans.* $9.93817 - 10$. **12.** $\log 0.08673$.
13. $\log 0.3078$. **14.** $\log 0.003078$. **15.** $\log 0.03078$.

Given $\log 6.509 = 0.81351$ and $\log 468.3 = 2.67052$, find the value of N in each of the following cases.

16. $\log N = 2.81351$. **17.** $\log N = 0.67052$.
18. $\log N = 1.81351$. **19.** $\log N = 5.81351$.
20. $\log N = 8.67052$. **21.** $\log N = 8.67052 - 10$.
22. $\log N = 9.81351 - 10$. **23.** $\log N = 6.67052 - 10$.
24. $\log N = 8.81351 - 10$.

WRITTEN EXERCISES

Given $\log 2 = 0.30103$ and $\log 3 = 0.47712$, find the number of places to the left of the decimal point or the position of the first significant figure for each of the following numbers.

25. 2^{35}. *Ans.* $\log 2^{35} = 35 \log 2 = 35 \times .30103 = 10.53605$.

$\therefore$ There are 11 places to the left of the decimal point in 2^{35}.

26. 3^{24}. **27.** 6^{15}. *Ans.* 12 places.
28. $(1.5)^{27}$. **29.** $(4\frac{1}{2})^{14}$. *Ans.* 10 places.
30. $(20)^{12}$. **31.** $(300)^{5}$. *Ans.* 13 places.
32. $(0.2)^{11}$. *Ans.* $\log (0.2)^{11} = 11 \log (0.2)$.

$\therefore \log (0.2)^{11} = 11 \times (9.30103 - 10) = 102.31133 - 110 = 2.31133 - 10$.

$\therefore$ The first significant figure is in the 8th place to the right of the decimal point.

33. $(0.03)^{6}$. *Ans.* 10th place. **34.** $(0.06)^{7}$.

32. A table of logarithms. — Table I is a table of the mantissas of all whole numbers from 1 to 9999 correct to five decimal places. The mantissa of the logarithm of *any* four-figure number can, therefore, be found in this table, since the mantissa depends only on the sequence of significant figures in the number and not on the position of the decimal point. This economy in the preparation and use of tables is one of the advantages gained by using 10 as the base of logarithms.

In seeking the logarithm of a number we look for the first three significant figures of the number in the column N at the left of the page. This gives the row in which the mantissa is to be found. The column is that in which the fourth figure of the number appears at the top of the page. The entry in the table gives only the last three figures of the mantissa. The first two figures, to save space, are printed only once, in the column headed 0 and at the left. Any entry is to have prefixed to it the last preceding pair of numbers at the left of the 0 column, unless it is marked with an asterisk, in which case the next following pair is to be used. The characteristic is determined by the position of the decimal point.

Example 1. Find log 16.74. We find 167 in column N, page 3. In the column 4 we find the entry 376 without an asterisk. The mantissa is therefore .22376. The characteristic is 1. Therefore log 16.74 = 1.22376.

Example 2. Find log .06763. On page 13, line 676, column 3, we find the entry *014. The mantissa is therefore .83014, and log .06763 = 8.83014 − 10.

33. The antilogarithm. — If $x = \log N$, N is said to be the *antilogarithm* of x, $N = \text{antilog } x$. In order to find the antilogarithm

of a given logarithm we seek the given mantissa in the body of the table. The first three figures of the antilogarithm then appear in the same line in the column N, and the fourth figure appears at the top of the page. The decimal point is then placed by the value of the characteristic.

EXAMPLES. 1. Find x, given $\log x = 2.84036$. We seek the mantissa .84036 in the body of the table. It is on page 13. The corresponding figures are 6924. Since the characteristic is 2, $x = 692.4$.

2. Find x, given $\log x = 8.90157 - 10$. On page 15 we find $x = .07972$.

3. Find antilog 0.46045. *Ans.* 2.887.

EXERCISES

Find the logarithms of the following numbers.

1.	87130.	*Ans.*	4.94017.	**2.**	45.62.
3.	0.06209.	*Ans.*	8.79302 − 10.	**4.**	1.280.
5.	100.3.	*Ans.*	2.00130.	**6.**	.002615.
7.	.2457.	*Ans.*	9.39041 − 10.	**8.**	3637.
9.	.6166.	*Ans.*	9.79000 − 10.	**10.**	7.769.
11.	1.087.	*Ans.*	0.03623.	**12.**	632.8.

Find the antilogarithms of the following numbers.

13.	1.82406.	*Ans.*	66.69.	**14.**	2.94275.
15.	9.91955 − 10.	*Ans.*	0.8309.	**16.**	8.76916 − 10.
17.	4.83948.	*Ans.*	69100.	**18.**	7.63347 − 10.
19.	2.69011.	*Ans.*	489.9.	**20.**	3.82014.
21.	0.00087.	*Ans.*	1.002.	**22.**	9.99029 − 10.
23.	5.25018.	*Ans.*	177900.	**24.**	2.62003.

34. Interpolation. — The logarithm of a five-figure number can be found from the table by interpolation. (Article **17**.)

Before seeking the logarithm of any number of more than five significant figures, we will generally round it off to the nearest five-figure number, for our tables are not designed to give greater accuracy than this.

EXAMPLE 1. Find log 37.826. On page 7 of the tables we find

$$\begin{array}{rr} \log 37.82 = & 1.57772 \\ \log 37.83 = & 1.57784 \\ \hline \text{Tabular difference} = & 12. \end{array}$$

Therefore, in the last place we add $.6 \times 12$ or 7 to log 37.82, getting log 37.826 = 1.57779.

The work of interpolation is made easier, however, by the tables of proportional parts which are at the right of each page. The numbers above the short columns are the tabular differences for the page. The nine numbers in a column are, in order, .1, .2, .3, etc. of the number above the column. Thus, on page 5, we see that .7 of the tabular difference 17 is 11.9.

EXAMPLE 2. Find log 152.47. On page 3 we find log 152.4 = 2.18298 and log 152.5 = 2.18327, with tabular difference 29. In the proportional parts column for 29, we find $.7 \times 29 = 20.3$. Therefore log 152.47 = 2.18318.

EXAMPLE 3. Find log .010535.
From page 2, tabular difference = 41.
Proportional part = $.5 \times 41 = 20.5$.

If this is rounded off to 21 the result is even, and

$$\log .010535 = 8.02264 - 10.$$

EXAMPLE 4. Find x, given $\log x = 1.34935$.
From page 4, tabular difference = 19.

In the table of proportional parts for 19, we seek the nearest number to our difference, which is (35 − 28) or 7. We find 7.6, which corresponds to .4. Therefore $x = 22.354$.

EXERCISES

Find the logarithms of the following numbers.

1. 33.587.	*Ans.* 1.52617.	**2.** 114.34.
3. 1.6307.	*Ans.* 0.21237.	**4.** 4.8606.
5. 26354.	*Ans.* 4.42085.	**6.** 3.1667.
7. 53.399.	*Ans.* 1.72753.	**8.** 334.98.
9. .038096.	*Ans.* 8.58088 − 10.	**10.** .11295.
11. .11197.	*Ans.* 9.04910 − 10.	**12.** .025795.

Find the antilogarithms of the following numbers.

13.	1.57708.	*Ans.*	37.764.	**14.**	2.54089.
15.	9.18314 − 10.	*Ans.*	.15246.	**16.**	7.09821 − 10.
17.	3.21733.	*Ans.*	1649.4.	**18.**	1.11378.
19.	8.13008 − 10.	*Ans.*	.013492.	**20.**	2.04347.
21.	3.09216.	*Ans.*	1236.4.	**22.**	0.21103.
23.	0.20131.	*Ans.*	1.5897.	**24.**	1.15204.

35. Computation by logarithms. — In any problem of computation, before looking up logarithms one should make a blank outline for all of the work, indicating all of the operations. It is then possible to do the mechanical work without interruption, and thereby gain in speed and accuracy. Every part of the work, moreover, is labelled, which aids in later checking or examination.

EXAMPLE 1. Find the value of

$$x = 21.48 \times 306.4 \times .01206.$$

By Theorem I, Article **25**,

$$\log x = \log 21.48 + \log 306.4 + \log .01206.$$

The outline is as follows.

$$\begin{aligned} \log 21.48 &= \\ \log 306.4 &= \\ \log .01206 &= \quad\quad (+) \\ \log x &= \\ x &= \end{aligned}$$

We can fill this out from the tables, getting

$$\begin{aligned} \log 21.48 &= 1.33203 \\ \log 306.4 &= 2.48629 \\ \log .01206 &= 8.08135 - 10\ (+) \\ \log x &= 11.89967 - 10 \\ x &= 79.372 \end{aligned}$$

Here x is found as the antilogarithm of 1.89967.

EXAMPLE 2. Find the value of

$$x = \frac{38.614 \times 1.8263 \times 1.2345}{496.23 \times 0.26944}.$$

$$\begin{aligned}
\log 38.614 &= 1.58674 \\
\log 1.8263 &= 0.26157 \\
\log 1.2345 &= 0.09150 \quad (+) \\
\log \text{numerator} &= 11.93981 - 10 \\
\log \text{denominator} &= 2.12614 \quad (-) \\
\log x &= 9.81367 - 10 \\
x &= 0.65113.
\end{aligned}$$

$$\begin{aligned}
\log 496.23 &= 2.69568 \\
\log 0.26944 &= 9.43046 - 10 \quad (+) \\
\log \text{denominator} &= 12.12614 - 10
\end{aligned} \leftarrow$$

It will be noticed that 10 was added to and subtracted from the logarithm of the numerator. This was done to avoid subtracting 2.12614 from a smaller number.

EXAMPLE 3. Find the value of

$$x = \sqrt[3]{\frac{28.693 \times (2.3614)^2}{89.325 \times 11.283}}.$$

By Article **25**, we have

$$\log x = \tfrac{1}{3}[\log 28.693 + 2 \log 2.3614 - \log 89.325 - \log 11.283].$$

The outline is:

$$\begin{aligned}
\log 2.3614 &= \\
&\quad \times 2 \\
\log (2.3614)^2 &= \\
\log 28.693 &= \quad (+) \\
\log \text{numerator} &= \\
\log \text{denominator} &= \quad (-) \\
\log \text{radicand} &= \quad (\div 3 \\
\log x &= \qquad \rightarrow x =
\end{aligned}$$

$$\begin{aligned}
\log 89.325 &= \\
\log 11.283 &= \quad (+) \\
\log \text{denominator} &= \leftarrow
\end{aligned}$$

This becomes:

$$\begin{aligned}
\log 2.3614 &= 0.37317 \\
&\quad \times 2 \\
\log (2.3614)^2 &= 0.74634 \\
\log 28.693 &= 1.45778 \quad (+) \\
\log \text{numerator} &= 12.20412 - 10 \\
\log \text{denominator} &= 3.00340 \quad (-) \\
\log \text{radicand} &= 29.20072 - 30 \quad (\div 3 \\
\log x &= 9.73357 - 10, \qquad \rightarrow x = 0.54146.
\end{aligned}$$

$$\begin{aligned}
\log 89.325 &= 1.95098 \\
\log 11.283 &= 1.05242 \quad (+) \\
\log \text{denominator} &= 3.00340
\end{aligned} \leftarrow$$

Here, before dividing 9.20072 − 10 by 3, which would have given the awkward number (3.06691 − 3.33333), we added and subtracted 20, making the negative part 30. In general, *before dividing a negative logarithm we make its negative part an exact multiple of the divisor by adding and subtracting a suitable number.*

EXERCISES

Find by logarithms the value of each of the following numbers, correct to five significant figures.

1. 48.32×9876. *Ans.* 477210.
2. 506.3×8.503.
3. $619.2 \times .04598$. *Ans.* 28.471.
4. $23.79 \times .08395$.
5. $83.69 \div 7.631$. *Ans.* 10.967.
6. $453.3 \div 6.029$.
7. $.9582 \div .03644$. *Ans.* 26.295.
8. $.6307 \div .05824$.
9. $73.06 \div 84.11$. *Ans.* .86862.
10. $3.086 \div 51.22$.
11. $.01699 \div .2525$. *Ans.* .067287.
12. $.04404 \div .07329$.
13. $\dfrac{24.68 \times 93.21}{5.040 \times 11.63}$. *Ans.* 39.246.
14. $\dfrac{54.23 \times 864.2}{918.2 \times 1.001}$.
15. $\dfrac{2.821 \times .06498}{54.26 \times .9801}$. *Ans.* .0034469.
16. $\dfrac{30.41 \times 2.613}{6 \times 87.36}$.
17. $(1.623)^5$. *Ans.* 11.262.
18. $(5.321)^6$.
19. $(.3647)^4$. *Ans.* .017691.
20. $(.06319)^5$.
21. $\sqrt{47980}$. *Ans.* 219.04.
22. $\sqrt{689700}$.
23. $\sqrt[3]{12.34}$. *Ans.* 2.3108.
24. $\sqrt[3]{702.3}$.
25. $\sqrt[5]{1.251}$. *Ans.* 1.0458.
26. $\sqrt[4]{119.6}$.
27. $\sqrt[3]{.03579}$. *Ans.* .32955.
28. $\sqrt[3]{.6745}$.
29. $\sqrt[6]{.4444}$. *Ans.* .87358.
30. $\sqrt[11]{.006254}$.
31. $\dfrac{3.142 \times (10.48)^2 \times 9.05}{72.93 \times \sqrt[3]{5.146}}$. *Ans.* 24.803.
32. $\dfrac{(5.021)^3 \times 11.38}{\sqrt{36.37} \times \sqrt[3]{4.021}}$.
33. $\left(\dfrac{16.11 \times \sqrt{2.049}}{3.108 \times 10.74}\right)^5$. *Ans.* .15738.
34. $\left(\dfrac{\sqrt{23.49} \times 6.01}{\sqrt{4568} \times 21.11}\right)^{12}$.
35. $\sqrt[3]{\dfrac{40.32 \times (.5114)^2}{16.91 \times 200.3}}$. *Ans.* .14602.
36. $\sqrt[7]{\dfrac{.8543 \times 20.18}{69.69 \times 56.01}}$.
37. 24.638×11.483. *Ans.* 282.91.
38. $.57693 \times 9.0468$.
39. $48.623 \div 1.4021$. *Ans.* 34.678.
40. $.38497 \div 6.7604$.
41. $\dfrac{.57916 \times 1102.1}{407.23 \times (16.927)^2}$. *Ans.* .0054704.
42. $\dfrac{(1.2616)^2 \times 11.982}{(3.1313)^5}$.
43. $\sqrt[3]{315.65}$. *Ans.* 6.8088.
44. $\sqrt[3]{3156.5}$.

45. $\sqrt[3]{.015995}$. *Ans.* .25196. **46.** $\sqrt[3]{.0022795}$.

47. $\sqrt{\dfrac{105.23}{51.688\times 11.963}}$. *Ans.* .41254. **48.** $\sqrt[3]{\dfrac{.098623\times 11.003}{61.107}}$.

49. $(.026549)^{\frac{5}{3}}$. *Ans.* .0023627. **50.** $(.18896)^{\frac{4}{7}}$.

Find the values of the following expressions for the indicated values of the quantities involved. In Examples **51** to **54**, $k = 6.5579\times 10^{-8}$.

51. $F = \dfrac{kmm'}{d^2}$, when $m = 5837$, $d = 24{,}378$, $m' = 11{,}432$.

Ans. $F = 7.3635 \times 10^{-9}$.

52. $F = \dfrac{kM}{r^2}$, when $M = 1372.5$, $r = 15.82$.

53. $F = \dfrac{2kM}{r}$, when $M = 4.87$, $r = 372.8$.

Ans. $F = 1.7133 \times 10^{-9}$.

54. $W = \dfrac{4kM^2}{7a}$, when $a = 24.76$, $M = 3789$.

55. $f = \dfrac{Mv^2}{R}$, when $M = 49.82$, $v = 5563$, $R = 17.94$.

Ans. $f = 8.594 \times 10^7$.

56. $T = 2\pi\sqrt{\dfrac{I}{Mgh}}$, when $g = 980.6$, $I = 53{,}933$, $h = 4.32$, $M = 278.4$.

57. $s = \frac{1}{2}\,gt^2$, when $g = 980.6$, $t = .8$. *Ans.* $s = 313.79$.

58. $f = \dfrac{6\pi^2\gamma^2a^2b^2x}{(a^2+x^2)^{\frac{5}{2}}}$, when $a = 15.3$, $b = .873$, $x = 7.894$, $\gamma = .0376$.

59. $n = \dfrac{8\pi Mk^2l}{a^4\tau^2}$, when $\tau = 7$, $l = 200$, $M = 5247$, $k = 2.24$, $a = .762$.

Ans. $n = 8010800$.

60. $q = \dfrac{2\pi kt(V_a - V_b)}{2.3026\,(\log_{10} b - \log_{10} a)}$, when $t = 5$, $k = .97$, $a = 6$, $b = 9$, $V_a = 477$, $V_b = 382$.

Historical note: Logarithms were invented by the Scotchman, John Napier, Baron of Merchiston (1550–1617). He published a partial account of his dis-

coveries in 1614. Merchiston Castle, now within the town of Edinburgh, at present houses a school for boys, the "dux" or head boy of which is said to be allowed the use of Napier's room as a study. The use of 10 as a base was suggested by the Englishman Briggs, a disciple of Napier. Briggs also gave the first tables of logarithms to the base 10.

***36. Cologarithms.** *The cologarithm of a number is the logarithm of the reciprocal of the number:*

$$\operatorname{colog} x = \log\left(\frac{1}{x}\right) = -\log x.$$

EXAMPLE. $\operatorname{colog} 4.6723 = \log\left(\frac{1}{4.6723}\right) = -\log 4.6723 = -.66953.$

In order to write this with a positive mantissa we add and subtract 10, getting

$$\operatorname{colog} 4.6723 = (10 - .66953) - 10 = 9.33047 - 10.$$

By using the following rule we can make the necessary subtraction, and so write down the cologarithm, without first writing down the logarithm: *To subtract a number (less than 10) from 10, begin at the left and subtract each figure from 9 except the last, and subtract the last figure from 10.*

In computing a fraction it is often convenient to add the cologarithms of the factors of the denominator to the logarithms of the factors of the numerator, rather than to subtract the sum of their logarithms. In the later parts of this book solutions will be given without the use of cologarithms. It is suggested, however, that the student will frequently find it desirable to use cologarithms in the solutions of similar problems.

EXAMPLE 1. $\operatorname{colog} 274.61 = 7.56128 - 10.$

EXAMPLE 2. Find colog .076432.

$$\begin{aligned}\operatorname{colog} .076432 &= -\log (.076432)\\ &= -(8.88327 - 10) = 10 - 8.88327\\ &= 1.11673.\end{aligned}$$

EXAMPLE 3. Find $N = \dfrac{27.468 \times 59.742}{3.8941 \times 47.631}$.

(*a*) *Solution without cologarithms.*

$$\begin{aligned} \log 27.468 &= 1.43883 \\ \log 59.742 &= 1.77628 \ (+) \\ &\overline{\quad 3.21511} \\ &\quad 2.26830 \ (-) \\ \log N &= 0.94681 \\ N &= 8.8472. \end{aligned}$$

$$\begin{aligned} \log 3.8941 &= 0.59041 \\ \log 47.631 &= 1.67789 \ (+) \\ &\overline{\quad 2.26830} \end{aligned}$$

(*b*) *Solution using cologarithms.*

$$\begin{aligned} \log 27.468 &= 1.43883 \\ \log 59.742 &= 1.77628 \\ \operatorname{colog} 3.8941 &= 9.40959 - 10 \\ \operatorname{colog} 47.631 &= 8.32211 - 10 \ (+) \\ \log N &= 20.94681 - 20 \\ N &= 8.8472. \end{aligned}$$

EXERCISES

Write the cologarithms of the following numbers, without first writing the logarithms.

1. 84.63.	*Ans.*	8.07248 − 10.	**2.** 96.87.
3. 581.3.	*Ans.*	7.23560 − 10.	**4.** 443.1.
5. 4.679.	*Ans.*	9.32985 − 10.	**6.** 8.319.
7. 0.04293.	*Ans.*	1.36724.	**8.** 0.004427.
9. 0.3634.	*Ans.*	0.43962.	**10.** 0.5757.
11. 21.683.	*Ans.*	8.66388 − 10.	**12.** 704.57.
13. 6.0278.	*Ans.*	9.21984 − 10.	**14.** 7.2486.
15. 356.97.	*Ans.*	7.44737 − 10.	**16.** 22.796.
17. 25.795.	*Ans.*	8.58846 − 10.	**18.** 1179.5.
19. 0.0046797.	*Ans.*	2.32978.	**20.** 0.088888.

Find the reciprocals of the following numbers, using cologarithms.

21. 78.463.	*Ans.*	0.012745	**22.** 532.67.
23. 1.3627.	*Ans.*	0.73383.	**24.** 2.4765.
25. 0.026798.	*Ans.*	37.316.	**26.** 0.0046193.

Find the values of the following numbers, using cologarithms.

27. $\dfrac{56.72 \times 1.362}{71.69 \times 0.3728}$. *Ans.* 2.8905.

28. $\dfrac{397.4 \times 0.04625}{6.150 \times 1.703}$.

29. $\sqrt[3]{\dfrac{6.0783 \times .21667}{146.02 \times .087119}}$. *Ans.* .46954

30. $\left(\dfrac{.028617 \times 23.346}{508.19 \times .11045}\right)^{\frac{3}{4}}$.

CHAPTER IV

THE RIGHT TRIANGLE

37. Logarithms of trigonometric functions. — In a right triangle, ABC, let us suppose that $a = 243.4$, $\alpha = 38° 42'$, and that we wish to find the hypotenuse, c. By Article **10**, we have

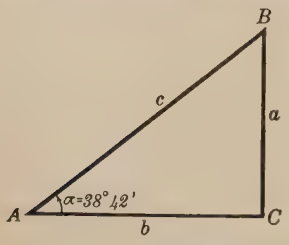

Fig. 25

$$c = \frac{a}{\sin \alpha} = \frac{243.4}{\sin 38° 42'},$$

and, using Table III to find $\sin 38° 42'$, we get

$$c = \frac{243.4}{.62524}.$$

We perform the division by logarithms:

$$\begin{aligned} \log 243.4 &= 12.38632 - 10 \\ \log .62524 &= 9.79605 - 10 \quad (-) \\ \log c &= 2.59027 \\ c &= 389.29 \text{ or } 389.3 \text{ to four places.} \end{aligned}$$

It is unnecessary, however, first to look up $\sin \alpha$ and then its logarithm, if we use Table II, which gives the logarithms of the trigonometric functions directly. On page 60, we find $\log \sin 38° 42' = 9.79605 - 10$.

The sine or cosine of any angle is less than 1. So also is the tangent of any angle between 0° and 45° or the cotangent of any angle between 45° and 90°. Consequently the logarithms of such functions are negative. Therefore *the number* -10 *should be appended to any entry in the first, second or fourth principal column of Table II.* The cotangent of an angle less than 45° (or the tangent of an angle between 45° and 90°) is greater than 1; its logarithm is therefore

positive, *and* -10 *should not be appended to any entry in the third principal column.*

Interpolation to the nearest tenth of a minute may be carried out by the methods already explained (Articles **17, 18, 19, 33**). Greater accuracy should not be sought in a five-place table. In Table II, the tabular differences are printed in the columns marked *d* for Log Sin and Log Cos, and in the column marked *cd* (common difference) for Log Tan and Log Cot.

EXAMPLE 1. Find log sin 8° 17.3′. On page 30, we find

$$\log \sin 8° 17' = 9.15857 - 10,$$

with tabular difference 87 in the *d* column. From the table of proportional parts for 87, we find $.3 \times 87 = 26.1$. Therefore $\log \sin 8° 17.3' = 9.15883 - 10$.

EXAMPLE 2. Find log tan 84° 28.8′. On page 27, we find $\log \tan 84° 28' = 1.01378$ with tabular difference 132. Since $.8 \times 132 = 105.6$, $\log \tan 84° 28.8' = 1.01378 + .00106 = 1.01484$.

EXAMPLE 3. Find θ, given $\log \cos \theta = 9.51797 - 10$. On page 41, we find $\log \cos 70° 45' = 9.51811 - 10$, with tabular difference 37. Our difference is $811 - 797 = 14$. From the table of proportional parts for 37, we see that 14 corresponds most nearly to .4. Therefore $\theta = 70° 45.4'$.

EXERCISES

Find

1. log sin 9° 38′.	*Ans.* 9.22361 − 10.	**2.** log sin 47° 46′.
3. log cos 48° 49′.	*Ans.* 9.81854 − 10.	**4.** log cos 23° 15′.
5. log tan 26° 22′.	*Ans.* 9.69520 − 10.	**6.** log tan 75° 43′.
7. log cot 54° 28′.	*Ans.* 9.85380 − 10.	**8.** log cot 31° 24′.
9. log sin 18° 18.7′.	*Ans.* 9.49719 − 10.	**10.** log sin 23° 44.8′.
11. log sin 63° 52.4′.	*Ans.* 9.95319 − 10.	**12.** log sin 72° 31.6′.
13. log cos 41° 12.5′.	*Ans.* 9.87640 − 10.	**14.** log cos 16° 52.5′.
15. log cos 72° 19.4′.	*Ans.* 9.48236 − 10.	**16.** log cos 66° 16.8′.
17. log tan 21° 39.8′.	*Ans.* 9.59902 − 10.	**18.** log tan 8° 14.6′.
19. log tan 52° 22.6′.	*Ans.* 0.11309.	**20.** log tan 69° 43.5′.
21. log cot 9° 28.7′.	*Ans.* 0.77740.	**22.** log cot 23° 18.9′.
23. log cot 55° 51.5′.	*Ans.* 9.83130 − 10.	**24.** log cot 48° 32.4′.

Find the value of θ, given:

25. $\log \sin \theta = 9.94773 - 10.$ *Ans.* 62° 27′.
26. $\log \sin \theta = 9.52598 - 10.$
27. $\log \cos \theta = 9.99822 - 10.$ *Ans.* 5° 11′.
28. $\log \cos \theta = 9.82509 - 10.$
29. $\log \tan \theta = 0.20562.$ *Ans.* 58° 5′.
30. $\log \tan \theta = 9.88707 - 10.$
31. $\log \cot \theta = 0.45610.$ *Ans.* 19° 17′.
32. $\log \cot \theta = 8.67267 - 10.$

33. $\log \sin \theta = 9.10909 - 10.$ *Ans.* 7° 23.2′.
34. $\log \sin \theta = 9.26301 - 10.$
35. $\log \sin \theta = 9.96227 - 10.$ *Ans.* 66° 27.7′.
36. $\log \sin \theta = 9.87318 - 10.$
37. $\log \cos \theta = 9.86514 - 10.$ *Ans.* 42° 51.4′.
38. $\log \cos \theta = 9.96836 - 10.$
39. $\log \cos \theta = 9.33724 - 10.$ *Ans.* 77° 26.6′.
40. $\log \cos \theta = 8.97693 - 10.$
41. $\log \tan \theta = 8.99302 - 10.$ *Ans.* 5° 37.2′.
42. $\log \tan \theta = 9.23016 - 10.$
43. $\log \tan \theta = 0.32141.$ *Ans.* 64° 29.7′.
44. $\log \tan \theta = 0.50238.$
45. $\log \cot \theta = 0.78560.$ *Ans.* 9° 18.3′.
46. $\log \cot \theta = 0.33975.$
47. $\log \cot \theta = 9.91833 - 10.$ *Ans.* 50° 21.3′.
48. $\log \cot \theta = 9.57600 - 10.$

49. Prove that the tabular difference for $\log \tan \theta$ must always be the same as for $\log \cot \theta$, so that both may be entered in a table of "common differences."

*__50.__ Find the cologarithms of the functions in Examples **1** to **24**.

In each case obtain the value of x, by logarithms.

51. $x = 74.62 \times \sin 16° 24'.$ *Ans.* 21.07.
52. $x = .05139 \times \tan 43° 16'.$

* This exercise is to be done only by those who have studied cologarithms.

53. $x = 524.1 \div \cos 58° 12'$. *Ans.* 994.6.
54. $x = 63.24 \div \sin 67° 42'$.
55. $x = 568.49 \times \cot 14° 11.7'$. *Ans.* 2247.4.
56. $x = .0046926 \times \tan 81° 14.2'$.
57. $x = 783.29 \div \cos 19° 23.7'$. *Ans.* 830.42.
58. $x = 8.0036 \div \sin 20° 2.4'$.
59. $x = \sqrt{6.2309 \times \sin 29° 14.6'}$. *Ans.* 1.7447.
60. $x = \sqrt[3]{.53674 \times \cos 72° 11.7'}$.

In each case find the value of θ by logarithms, given:

61. $\sin \theta = 14.763 \div 20.482$. *Ans.* 46° 7.2'.
62. $\cos \theta = 2.4862 \div 5.3294$.
63. $\tan \theta = 42.365 \div 31.497$. *Ans.* 53° 22.2'.
64. $\tan \theta = 8.6294 \div 11.636$.
65. $\cot \theta = 16.422 \div 13.115$. *Ans.* 38° 36.7'.
66. $\cot \theta = .94625 \div 2.1441$.

38. The accuracy of a computed result. Errors due to inaccuracies in the data. — The results of most problems in applied mathematics which involve computation are liable to certain errors. Apart from sheer mistakes, which need not be tolerated, these errors are of two sorts: errors due to inaccuracies in the data and errors due to limitations of the tables used. In the very nature of things these errors cannot be eliminated, but one can see to it that the results be *sufficiently accurate for the purpose in view.*

The data of most problems are the result of observations or measurements. These measurements are made with greater or less accuracy, depending on the purposes for which they are made and the difficulties to be overcome. The number of significant figures in the data should correspond to the accuracy with which measurements have been made.

To say that the distance by railroad from Milwaukee to Boston is 1183 miles means that the distance has been measured so accurately that it can safely be said to be nearer to 1183 miles than to 1182 miles or 1184 miles, or, in other words, that the distance is between 1182.5 miles and 1183.5 miles. If the bore of an

engine cylinder is given as 2.69 inches in diameter, then it is between 2.685 and 2.695; if it is given as 2.690 inches, then it is between 2.6895 inches and 2.6905 inches. In this connection we should notice that a zero placed at the end of a number sometimes indicates the degree of accuracy with which the number is given. There is, for example, a difference in implication in the numbers 27, 27.0, and 27.00. The first number presumably is accurate to the nearest unit and is given with "two-place" accuracy. The second is correct to the nearest tenth of a unit and has three-place accuracy. And the third has four-place accuracy.

Since the data of a problem are in general only approximate, results obtained from them by any process of computation will be only approximately correct. And the accuracy of the result will not exceed that of the data. For this reason the number of significant digits in a result should be rounded off so as not to exceed that of the data. Otherwise one would claim for the result an accuracy exceeding that of the data.

In applying this rule, one should keep in mind the geometrical or physical interpretation of a result. For example, if the two perpendicular sides of a right triangle are given as 6.41 feet and 8.62 feet, the hypotenuse may be computed to be 10.74 feet. It is proper to give the result in this form, though it contains one more significant figure than did the data, for the numbers all represent distances to the nearest hundredth of a foot. Only in rare cases, in such problems, will the last figure be incorrect.

Sometimes it is puzzling to know what degree of accuracy in one kind of unit corresponds to a certain accuracy in the other. In triangles it is proper to suppose that *four-place accuracy in the sides corresponds to accuracy to the nearest minute in the angles; five-place accuracy in the sides corresponds to accuracy to the nearest tenth of a minute in the angles.*

39. Errors due to the limitations of the tables. — Many problems in applied mathematics contain no errors in the data. This is true, for example, in most problems of compound interest, where the principal, rate, time, etc. are known exactly. But even in such problems results are subject to certain errors due to the use of tables. A logarithm or a sine is usually a never-ending decimal which can be

given in a table only to a certain number of decimal places, that is, approximately but not exactly.

Five-place accuracy in results can usually be had by using five-place tables; four-place accuracy with four-place tables, etc., although errors may enter into the last place of the results. At any rate, one must not expect five-place accuracy when using four-place tables. On the other hand, to use five-place tables when two or three places would suffice involves unnecessary labor. But the use of tables having one more place than will be retained obviates unnecessary errors and, by doing away with interpolation, probably saves as much time as it wastes.

40. Logarithmic solution of the right triangle. —

EXAMPLE. Solve the right triangle ABC, given

$$a = 417.63, \quad b = 368.47.$$

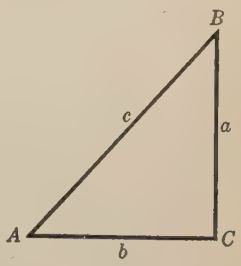

FIG. 26

We use the formulas

$$\tan \alpha = \frac{a}{b}, \quad c = \frac{a}{\sin \alpha}, \quad \beta = 90° - \alpha.$$

$$\begin{aligned} \log 417.63 &= 2.62079 \\ \log 368.47 &= 2.56640 \quad (-) \\ \log \tan \alpha &= .05439 \\ \alpha &= 48° 34.7' \\ \beta &= 90° - \alpha = 41° 25.3'. \end{aligned}$$

$$\begin{aligned} \log 417.63 &= 12.62079 - 10 \\ \log \sin \alpha &= 9.87498 - 10 \quad (-) \\ \log c &= 2.74581 \\ c &= 556.94. \end{aligned}$$

Check: The formula

$$c^2 = a^2 + b^2$$

is not suitable, as it stands, for checking by logarithms. We can write it

$$a^2 = c^2 - b^2,$$

or

$$a^2 = (c + b)(c - b).$$

Therefore, as a check, we should find that

$$2 \log a = \log (c + b) + \log (c - b).$$

This is the best check for the solution of a right triangle by logarithms.

In the present example,

$$c + b = 925.41, \qquad \log (c + b) = 2.96634$$
$$c - b = 188.47, \qquad \log (c - b) = 2.27524 \quad (+)$$
$$\log [(c + b)(c - b)] = 5.24158.$$

And $\log a = 2.62079$,
so that $\log a^2 = 2 \log a = 5.24158$,

which, in this case, checks exactly.

EXERCISES

Using logarithms solve the right triangle for each of the following sets of data. Check the solution.

1. $a = 27.46, c = 38.82$. *Ans.* $\alpha = 45° 1', \beta = 44° 59', b = 27.44.$
2. $a = 692.7, c = 1010.0.$
3. $b = 5046, c = 8423$. *Ans.* $\alpha = 53° 12', \beta = 36° 48', a = 6744.$
4. $b = 126.3, c = 142.5.$
5. $a = 81.48, b = 72.19$. *Ans.* $\alpha = 48° 28', \beta = 41° 32', c = 108.86.$
6. $a = 1246, b = 1583.$
7. $a = 14.92, \alpha = 65° 3'$. *Ans.* $\beta = 24° 57', b = 6.94, c = 16.46.$
8. $a = 106.6, \alpha = 17° 18'.$
9. $b = 1.927, \alpha = 40° 0'$. *Ans.* $\beta = 50° 0', a = 1.617, c = 2.516.$
10. $b = 0.7078, \alpha = 63° 21'.$
11. $a = 551.45, b = 490.13.$
Ans. $\alpha = 48° 22.2', \beta = 41° 37.8', c = 737.78.$
12. $a = 12.138, b = 18.634.$
13. $c = 177.64, a = 161.48.$
Ans. $\alpha = 65° 22.3', \beta = 24° 37.7', b = 74.028.$
14. $c = 0.70385, b = 0.52664.$
15. $c = 9.0024, b = 3.6512.$
Ans. $\alpha = 66° 4.4', \beta = 23° 55.6', a = 8.2287.$
16. $c = 0.064493, a = 0.036254.$

17. $a = 83.577, \alpha = 71° 12.4'$.

Ans. $\beta = 18° 47.6', b = 28.441, c = 88.282$.

18. $a = 5.9528, \beta = 24° 16.3'$.

19. $b = 0.81325, \alpha = 31° 31.7'$.

Ans. $\beta = 58° 28.3', a = .49891, c = .95410$.

20. $b = 161.46, \beta = 38° 47.5'$.

21. $c = 38.506, \alpha = 49° 18.3'$.

Ans. $\beta = 40° 41.7', a = 29.195, b = 25.107$.

22. $c = 798.05, \alpha = 68° 16.1'$.

23. $c = 8345.6, \beta = 58° 47.6'$.

Ans. $\alpha = 31° 12.4', a = 4324.0, b = 7138.0$.

24. $c = 5.2869, \beta = 25° 31.0'$.

Solve the following oblique triangles, by dividing them into right triangles. Draw the figures.

25. $b = 243.4, \alpha = 31° 11', \beta = 47° 24'$.

(*Solution:* We draw the figure, and draw CD perpendicular to AB. Denote CD by h, AD by x, DB by y. We then use, in succession, the formulas:

$\gamma = 180° - (\alpha + \beta)$, $x = b \cos \alpha$, $h = b \sin \alpha$,
$y = h \cot \beta$, $c = x + y$, $a = h / \sin \beta$.

We find:

$\gamma = 101° 25'$, $x = 208.23$, $\log h = 2.10046$,
$y = 115.89$, $c = 324.1$, $a = 171.2$.

The student should perform the computation.)

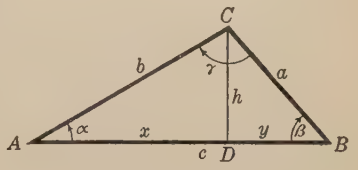

FIG. 27

26. $b = 78.95, \alpha = 24° 46', \beta = 39° 57'$.

27. $a = 10.89, \alpha = 41° 17', \beta = 46° 52'$.

Ans. $\gamma = 91° 51', b = 12.04, c = 16.50$.

28. $a = 479.4, \alpha = 26° 48', \beta = 50° 0'$.

29. $b = 71.39, \alpha = 42° 11', \beta = 116° 27'$.

Ans. $\gamma = 21° 22', a = 53.54, c = 29.05$.

30. $b = 59.46, \alpha = 108° 52', \beta = 50° 16'$.

31. $b = 23.69$, $c = 30.47$, $\alpha = 31° 15'$.

Ans. $\beta = 50° 16'$, $\gamma = 98° 29'$, $a = 15.98$.

32. $a = 60.04$, $c = 102.31$, $\beta = 47° 29'$.

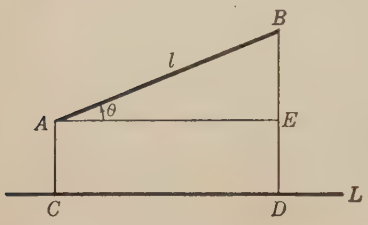

FIG. 28

***41. Projections.** — If C and D are the feet of the perpendiculars drawn from points A and B, respectively, to the line L, CD is called the *projection* of AB on L. From the figure,

$$CD = AE = l \cos \theta.$$

EXERCISES

In each of the following examples find the projections of AB on a line which makes the angle θ with AB and on a line perpendicular to such a line.

1. $AB = 367.24$, $\theta = 27° 18.7'$. *Ans.* 326.30, 168.50.

2. $AB = 982.36$, $\theta = 81° 47.4'$.

3. $AB = 4238.7$, $\theta = 60° 58.1'$. *Ans.* 2057.0, 3706.1.

4. $AB = 1026.4$, $\theta = 48° 16.4'$.

In each of the following examples find the length of the line which makes the angle θ with a line L, and whose projection on L is AB.

5. $AB = 986.3$, $\theta = 41° 52.7'$. *Ans.* 1324.7.

6. $AB = 1268.4$, $\theta = 68° 12.3'$.

7. $AB = 823.7$, $\theta = 79° 34.6'$. *Ans.* 4552.8.

8. $AB = 6.261$, $\theta = 32° 46.9'$.

***42. Surveyors' terms.** — *Inclination.* This is the angle which a line run on sloping ground makes with the horizontal. It is positive if the line rises in its forward direction and negative if it falls. In Figure 28, if AE is horizontal, θ is the inclination of AB.

Course of a line. This is the horizontal distance between the ends of the line.

In Figure 28, the course of AB is $AE = AB \cos \theta$. The course is the distance that appears on maps and descriptions of title to pieces of real estate.

Bearing of a line. This is the angle which the line makes with a north-south line.

It is 0° if the line runs due north or south, 90° if it runs exactly east or west. In writing it, give first the letter N or S, then the angle of deviation from the north or south, then the letter E or W. In Figure 29 the bearing of AB is N 30° E; that of AC is S 63° W.

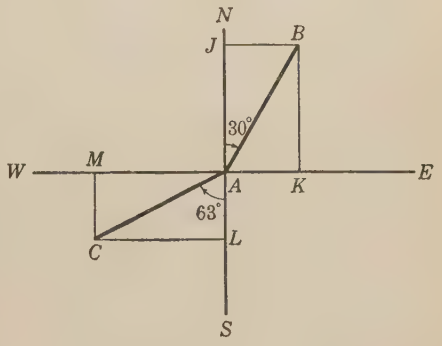

FIG. 29

Latitude of a course. This is the distance to the north (and then is positive) or to the south (and then is negative) which one goes in passing over the course.

In Figure 29, the latitude of AB is AJ. The latitude = length × cosine of bearing.

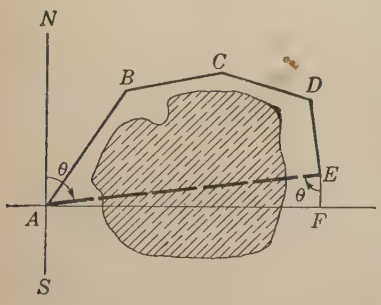

FIG. 30

Departure of a course. This is the distance to the east (and then is positive) or to the west (negative) which one goes while passing over the course.

In Figure 29, AK is the departure of course AB. The departure = length × sine of bearing.

The closing course. To avoid an obstacle which separates two points A and E, a surveyor measures the length and bearing of each

part of a broken line, such as $ABCDE$. Then AE is the closing course of the broken line. Its latitude and departure are the algebraic sums of the latitudes and departures, respectively, of the partial courses.

EXAMPLE 1. Find the course AF from the following data of lengths and inclinations.

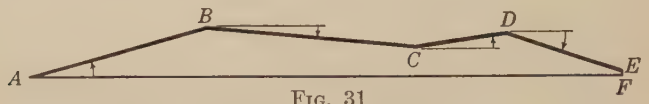

FIG. 31

Line	*Length (rods)*	*Inclination*	*(Solution) Course*
AB	95.27	$+15° 24'$	91.85
BC	109.32	$- 5° 16'$	108.86
CD	46.28	$+ 8° 13'$	45.80
DE	62.84	$-18° 43'$	59.52
AE			306.03 rods

For the course AB we have $95.27 \cos 15° 24' = 91.85$.
The other items in the last column are found in a similar way.
Show that point E is 1.63 rods higher than A.

EXAMPLE 2. Find the latitude, departure, length, and bearing of the closing course AE from the following data of lengths and bearings. (See Figure 30.) (Solution is in italics.)

Course	*Length (rods)*	*Bearing*	*Latitude*		*Departure*	
			+	−	+	−
AB	147.62	N 33° 12.3′ E	*123.52*		*80.84*	
BC	101.46	N 79° 28.4′ E	*18.54*		*99.75*	
CD	96.24	S 72° 34.1′ E		*28.83*	*91.82*	
DE	81.56	S 8° 14.4′ E		*80.72*	*11.69*	
Total			*142.06*	*109.55*	*284.10*	
AE	*285.94*	N 83° 28.3′ E	*32.51*		*284.10*	

For the course AB, $147.62 \times \cos 33° 12.3' = 123.52$,

and $147.62 \times \sin 33° 12.3' = 80.84$.

The other latitudes and departures are found in a similar way.

The latitude of $AE = 142.06 - 109.55 = 32.51$.

The bearing, θ, of the closing course AE is found by taking (from Figure 30)

$$\cot \theta = \text{latitude/departure} = 32.51/284.10;$$

from which $\theta = 83° 28.3'$. And the length of AE is

$$l = \text{latitude}/\cos \theta = 32.51/\cos 83° 28.3' = 285.94.$$

EXERCISES

Some of these exercises are too long for work in the classroom, but are suitable for home work.

From each of the following sets of data find the horizontal course of the line and its net rise or fall. Draw the figure.

1.

Line	*AB*	*BC*	*CD*	*DE*	*EF*
Length (rods)	76.39	28.24	53.67	102.31	37.59
Inclination	+11° 12′	−12° 43′	+9° 38′	−6° 51′	−14° 21′

Ans. Course 293.40 rods; net fall 3.92 rods.

2.

Line	*AB*	*BC*	*CD*	*DE*
Length (rods)	83.21	62.62	100.30	41.20
Inclination	−10° 16′	−7° 43′	+11° 54′	+5° 23′

3.

Line	*AB*	*BC*	*CD*	*DE*
Length (ft.)	241.3	98.6	101.3	294.7
Inclination	−7° 12′	+6° 24′	−14° 14′	+18° 33′

Ans. Course 715.0 ft.; net rise 49.6 ft.

4.

Line	*AB*	*BC*	*CD*	*DE*
Length (ft.)	143.7	91.3	114.5	83.7
Inclination	+6° 7′	+9° 24′	+5° 18′	−13° 39′

From each of the following sets of data find the latitude, departure, length, and bearing of the closing course. Draw the figure.

5.

Course	*AB*	*BC*	*CD*	*DE*	*EF*
Length (ft.)	68.6	95.1	207.4	103.8	96.8
Bearing	*N* 7° 0′ *W*	*N* 15° 51′ *E*	*N* 84° 41′ *E*	*S* 5° 7′ *E*	*S* 49° 53′ *W*

Ans. lat., +13.0; dep., +159.3; length, 159.8; *N* 85° 20′ *E*.

6.

Course	*AB*	*BC*	*CD*	*DE*
Length (ft.)	123.4	179.6	153.9	215.2
Bearing	*N* 73° 6′ *E*	*N* 22° 47′ *E*	*N* 84° 11′ *E*	*S* 54° 26′ *E*

From the following data find the latitude, departure, course, bearing, and net rise or fall of the closing course.

7.

Line	*AB*	*BC*	*CD*	*DE*	*EF*
Length (rods)	35.26	41.57	25.62	31.51	19.62
Inclination	$+10° 14'$	$+6° 27'$	$-5° 10'$	$-8° 49'$	$-11° 37'$
Bearing	$S\ 61° 10'\ E$	$N\ 81° 41'\ E$	$N\ 28° 18'\ E$	$N\ 32° 30'\ W$	$N\ 10° 18'\ E$

Ans. Lat., 57.00; dep., 70.00; course, 90.27; bearing, $N\ 50° 51'\ E$; net fall, 0.15 rods.

***43. Forces and vectors.** — In studying the forces that act on a body, it is convenient to think of them as vectors. A *vector* is a directed magnitude. Forces and velocities, for example, are vectors, since they have both magnitude and direction. A vector can be represented by a line segment whose length is proportional to the magnitude and whose direction is the direction of the vector. In Figure 32 the lines OP_1, OP_2, and OP_3 represent forces of three pounds, two pounds, and two and one-half pounds acting in the directions, $N\ 55° E$, $N\ 25° W$, and $S\ 20° E$.

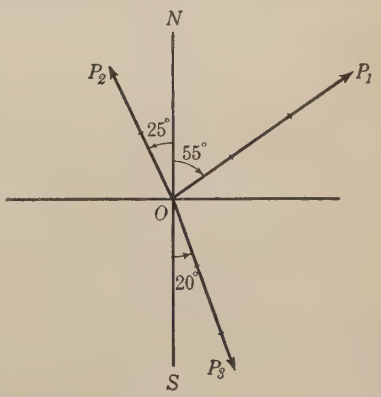

FIG. 32

If two or more forces act on a body at the same time, their combined effect is the same as would be that of a single force having a certain magnitude and direction. This single force is called the *sum* or *resultant* of the separate forces.

In a similar way, the velocity of a body may result from two or more elementary or component velocities. For example, if a man walks across the deck of a moving ship, his velocity relative to the ship combines with the velocity of the ship to give his velocity relative to the earth.

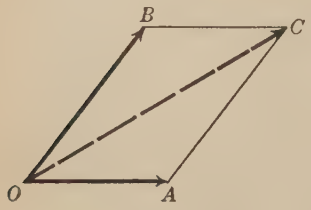

FIG. 33

If OA and OB represent two vectors, the sum or resultant of these vectors is represented both in direction and in magnitude by OC, the diagonal of the parallelogram of which OA and OB are adjacent sides.

It is a law of physics, proved both by experiment and theory, that *if two forces act on a body, they have the same effect as would their resultant acting alone.*

***44. Components of a force.** — If OC is the resultant of two forces, OA and OB, OA and OB are called *the components of OC in the directions OA and OB.* In particular, if the directions OA and OB are perpendicular to each other, and if OC makes the angle θ with OA,

$$OA = OC \cdot \cos\theta \quad \text{and} \quad OB = OC \cdot \sin\theta.$$

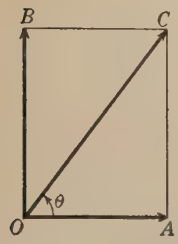

FIG. 34

EXAMPLE. Find the force necessary to push a ball which weighs 10 pounds up a plane inclined 20° to the horizontal.

The weight or force, OC, of ten pounds acts vertically downward. The component of this force OA, which is parallel to the inclined plane, and which the desired force must overcome, is

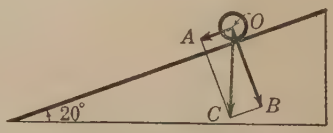

FIG. 35

$$OA = 10 \cos \angle AOC = 10 \cos 70° = 3.4 \text{ pounds.}$$

In order to roll the ball up hill it is therefore necessary to exert a force of 3.4 pounds in addition to the small force necessary to roll the ball on a level plane. The pressure of the ball against the plane is

$$OB = 10 \cos \angle BOC = 10 \cos 20° = 9.4 \text{ pounds.}$$

EXERCISES

1. A vertical force of 38.4 lbs. and a horizontal force of 17.6 lbs. act on a body. Find the magnitude and direction of their resultant.

Ans. 42.2 lbs., 65° 23′ from horizontal.

2. A body is acted on by a horizontal force of 98.3 lbs. and by a downward force of 173.6 lbs. Find the direction and magnitude of their resultant.

3. Find the horizontal and vertical components of a force of 987.6 lbs. inclined at 73° 25′ to the horizontal.

Ans. Horizontal, 281.9; vertical, 946.5.

4. Find the horizontal and vertical components of a force of 624.9 lbs. inclined at 21° 16′ to the horizontal.

5. A body is acted upon by a force of 706.3 lbs. in the direction N 53° 16′ E, and also by a force of 528.6 lbs. in the direction S 28° 39′ E. Find the east-west and north-south components, the direction and the magnitude of their resultant.

Ans. East, 819.5 lbs.; south 41.4; S 87° 6′ E; 820.5 lbs.

6. A body is acted upon by a force of 1263 lbs. in the direction N 42° 51′ E and by a force of 946 lbs. in the direction N 10° 18′ W. Find the east-west and north-south components, the direction, and the magnitude of the resultant.

7. A body is acted upon by a force of 1384 lbs. in the direction N 57° 12′ W, by a force of 1682 lbs., in the direction N 74° 46′ E, and by a force of 1964 lbs. in the direction S 6° 57′ W. Find the east-west and north-south components, the direction, and the magnitudes of their resultant.

Ans. East, 222 lbs.; south, 758 lbs.; S 16° 19′ E; 790 lbs.

8. A body is acted on by three forces having the following directions and magnitudes: 1006 lbs., N 17° 5′ E; 984 lbs., due east; 1382 lbs., S 31° 31′ W. Find the east-west and north-south components, the direction, and the magnitude of the resultant.

9. A body is acted on by a force of 18.63 lbs. in the direction N 28° 46′ E and by a force of 15.79 lbs. in the direction S 41° 28′ E. Find the magnitude and direction of a force which just counteracts them.

Ans. 19.93 lbs., S 76° 57′ W.

10. A certain force just counteracts the combined effect of a force of 213.6 lbs. in the direction $S\ 26°\ 29'\ E$ and a force of 187.7 lbs. in the direction $N\ 31°\ 11'\ W$. Find its magnitude and direction.

11. A ship travels due east making 20 miles per hour. A man walks directly across the deck at 4 miles per hour. Find his direction and velocity relative to the earth. *Ans.* $N\ 78°\ 41'\ E$; 20.4 mi./hr.

12. A duck which swims 5 miles per hour in still water swims directly across a current of 8 miles per hour. Find its actual velocity and direction.

13. An automobile is pulled by a cable up an inclined plane to the second floor of a repair shop. If the car weighs 2096 lbs. and the slope is 37° 45′ from the horizontal, what is the tension in the cable?

Ans. 1283 lbs.

14. The approach to a bridge is inclined at 14° 25′. What is the pressure against it due to the weight of a car which weighs 1895 lbs.?

15. How much more force is necessary to move a car weighing 2985 lbs. up a grade of 5° than to move one weighing 1985 lbs.?

Ans. 87.2 lbs. At a speed of 15 miles per hour this requires a difference in power of about 3.5 horse-power.

16. A mass of 387.46 lbs. rests on an inclined plane. A force of 238.18 lbs. is required to keep it from sliding down. What is the angle of inclination of the plane? Neglect friction.

17. A beam when horizontal can support 3847 lbs. What weight can it support if inclined at 28° 36′? *Ans.* 4382 lbs.

18. A steel rafter which is inclined at 47° 16′ supports a total weight of 4676 lbs. What is the end thrust or pressure of the rafter against the supporting girder?

19. A guy wire runs from the top of a telephone pole 38 ft. high to a point on the ground 23 ft. from the pole. The tension in the guy wire is 164 lbs. Find the horizontal component of this used in supporting the pole, and also the vertical component which adds to the downward thrust of the pole.

Ans. Horizontal, 84.9 lbs.; vertical, 140.3 lbs.

20. A cable is stretched from the top of a tower. At its point of attachment it makes an angle of 4° 2′ with the horizontal. Its

tension is 7846 lbs. Find its horizontal and vertical components.

21. A horizontal wire pulls with a force of 468 lbs. on the top of a pole 96 ft. high. Its pull is opposed by that of a guy wire attached to the ground at a point 44 ft. from the pole in the opposite direction. Find the tension in the guy wire if its pull just counteracts that of the horizontal wire. *Ans.* 1123 lbs.

22. A cable is supported by two steel towers. Its sag is such that at its point of support on one of the towers it makes an angle of 18° 19′ with the horizontal and its tension is 4980 lbs. Find its lateral pull and downward pressure on the tower.

23. A concrete pouring tower is 124 ft. high. Two guy wires run from its top to points on the ground which are on opposite sides of the tower, and are, respectively, 163 ft. and 71 ft. from the tower. The tension in the longer wire is 412 lbs. What is the tension in the shorter wire if it just overcomes the lateral pull of the longer one? What is the combined downward pull of the two wires on the tower? *Ans.* Tension, 660 lbs.; downward pull, 822 lbs.

24. A boat is fast on a bar. The crew run out an anchor 64 ft. from the boat. It sinks to a depth of 18 ft. below the level of the deck. By winding the anchor cable on a capstan on the deck the crew hope to free the boat. If a horizontal pull of 1874 lbs. is required, what tension will be necessary in the cable?

CHAPTER V

RADIAN MEASURE

45. Definition of a radian. — In most problems involving the numerical values of the angles of a triangle, it is customary to express angles in terms of degrees and minutes. This is not the only system of units for measuring angles, however, and for many purposes, notably in almost all problems involving the methods of calculus, it is best to express angles in terms of a different unit of angular measure called the radian.

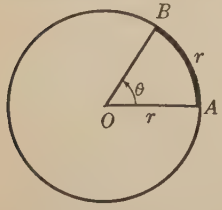

FIG. 36

Definition. A radian is an angle which, if its vertex is placed at the center of a circle, subtends on the circumference an arc equal to the radius of the circle.

In the figure, θ is equal to one radian, since the arc AB and the radius OA are equal. It is to be noticed that *a radian is an angle of fixed magnitude* and does not depend on the size of the circle at whose center it may happen to be placed, for the subtended arc is proportional to the radius.

46. Degrees and radians. — The circumference of any circle is equal to 2π times its radius. That is, the circumference contains 2π arcs each equal to the radius. Each such arc subtends at the center an angle of one radian, and the entire circumference subtends 2π radians. Therefore 360° equals 2π radians, or

(1) $$\pi \text{ radians} = 180°.$$

Consequently

(2) $$1 \text{ radian} = \frac{180}{\pi} = \frac{180}{3.1416} \text{ degrees},$$

or

(3) $$1 \text{ radian} = 57.2958° = 57° 17.75'.$$

On the other hand

$$1^\circ = \frac{\pi}{180} \text{ radians} = .017453 \text{ radians.} \tag{4}$$

When no other unit of angular measure is indicated it is assumed that an angle is expressed in radian measure. For example, in writing $\theta = \pi/2$ we mean that θ is $\pi/2$ radians or 90°.

EXERCISES

The following angles are given in radian measure. Express them in degrees.

1. $\pi/3$.	*Ans.* 60°.	**2.** 2π.
3. $\pi/4$.	*Ans.* 45°.	**4.** $\pi/6$.
5. $3\pi/2$.	*Ans.* 270°.	**6.** $2\pi/3$.
7. $2\pi/5$.	*Ans.* 72°.	**8.** $7\pi/6$.
9. $3\pi/4$.	*Ans.* 135°.	**10.** $11\pi/6$.
11. $4\pi/3$.	*Ans.* 240°.	**12.** $5\pi/3$.
13. $-5\pi/4$.	*Ans.* -225°.	**14.** $-\pi/10$.
15. $-\pi/2$.	*Ans.* -90°.	**16.** -2π.

Express in radian measure.

17. 36°.	*Ans.* $\pi/5$.	**18.** 90°.
19. 60°.	*Ans.* $\pi/3$.	**20.** 45°.
21. 330°.	*Ans.* $11\pi/6$.	**22.** -210°.
23. 120°.	*Ans.* $2\pi/3$.	**24.** 240°.
25. -135°.	*Ans.* $-3\pi/4$.	**26.** 150°.
27. 300°.	*Ans.* $5\pi/3$.	**28.** 270°.
29. 17°.	*Ans.* .2967.	**30.** 46°.
31. $-86^\circ\ 30'$.	*Ans.* -1.5097.	**32.** $113^\circ\ 15'$.
33. $24^\circ\ 16'$.	*Ans.* .42353.	**34.** $100^\circ\ 38'$.
35. $49^\circ\ 48.7'$.	*Ans.* .86937.	**36.** $62^\circ\ 29.4'$.

Give the values of the following functions.

37. $\sin \pi/6$.	*Ans.* 1/2.	**38.** $\cos \pi/3$.
39. $\tan \pi/4$.	*Ans.* 1.	**40.** $\cot \pi/6$.
41. $\cos 3\pi/4$.	*Ans.* $-\frac{1}{2}\sqrt{2}$.	**42.** $\tan 5\pi/4$.

Give in radian measure a value of θ less than 2π for each case:

43. $\sin\theta = \frac{1}{2}\sqrt{3}$. *Ans.* $\pi/3$.
44. $\tan\theta = \frac{1}{3}\sqrt{3}$.
45. $\tan\theta = -1$. *Ans.* $3\pi/4$.
46. $\sin\theta = 0$.
47. $\sec\theta = 2$. *Ans.* $\pi/3$.
48. $\cot\theta = 1$.

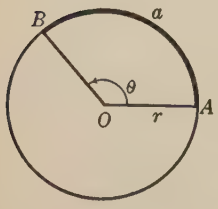

FIG. 37

47. Central angles and their subtended arcs. — If an angle equal to one radian is placed at the center of a circle of radius r, it subtends on the circumference an arc equal to r. Therefore a central angle equal to θ radians subtends on the circumference an arc whose length is equal to θr. If a denotes the length of the arc,

$$a = \theta r.$$

In this formula, we must remember, θ is *the number of radians* in the central angle. When we are given any two of the three numbers a, r, and θ, the formula may be used to find the third number.

EXAMPLE 1. On a circle of radius 9 inches, what arc will be subtended by a central angle of 36°? Here $r = 9$; the central angle is $36° = \pi/5$ radians, so that $\theta = \pi/5$. Therefore

$$a = \frac{\pi}{5} \cdot 9 = \frac{3.142 \times 9}{5} = 5.66 \text{ inches.}$$

EXAMPLE 2. On a circle 1 foot in diameter, what central angle will subtend an arc equal to 10 inches? If we use an inch as the unit of length, $r = 6$, $a = 10$, and

$$\theta = \frac{a}{r} = \frac{10}{6} = \frac{5}{3} \text{ radians} = \frac{5}{3} \times \frac{180°}{\pi} = 95° 30'.$$

EXAMPLE 3. What must be the radius of a wheel, on the rim of which an arc 16.1 inches long is subtended by a central angle of 25°? Here $\theta = 25\pi/180$, so that

$$r = \frac{a}{\theta} = \frac{16.1 \times 180}{25 \cdot \pi} = 36.9 \text{ inches.}$$

***48. Linear and angular velocity.** — If an object is moving uniformly, its *linear velocity* is its rate of motion; it is equal to the

distance passed over in a certain time divided by the time, or to the number of units of distance passed over in one unit of time.

EXAMPLE. An object which travels steadily 20 yards in 5 seconds has a linear velocity equal to $20/5 = 4$ yards per second.

If an object turns uniformly, its *angular velocity* is its rate of turning; it is equal to the angle through which it turns in a certain time divided by the time, or to the number of units of angle through which the object turns in one unit of time.

EXAMPLES. An object which turns uniformly through 150° in 10 seconds has an angular velocity equal to $150/10 = 15$ degrees per second. An object which turns through 12 radians in 3 minutes has an angular velocity equal to 4 radians per minute.

If, in t units of time, an object turns through the angle θ, its angular velocity is

$$\alpha = \frac{\theta}{t}. \tag{1}$$

Suppose that an object revolve uniformly about the point O with the angular velocity α radians per second. Let us find the linear velocity of a point P which is on the object at the distance r from O. In one second the line OP turns through α radians, and the point P moves through an arc whose length is $r \cdot \alpha$. The linear velocity, v, of the point P is therefore

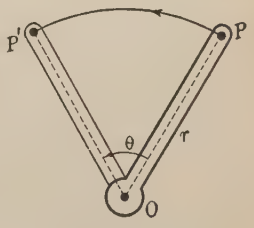

FIG. 38

$$v = r \cdot \alpha \tag{2}$$

units of length per second. From this and equation (1), we see that if OP turns uniformly through θ radians in t units of time, its linear velocity is

$$v = \frac{r \cdot \theta}{t}. \tag{3}$$

EXAMPLE 1. A pulley 1 foot in diameter makes 500 revolutions per minute. Find its angular velocity in radians per second and the speed of the belt which

drives the pulley. *Solution:* In each revolution the pulley turns through 2π radians, so that in one minute it turns through $500 \times 2\pi$ radians. By equation (1) its angular velocity is therefore

$$\alpha = \frac{1000\,\pi}{60} = 52.36 \text{ radians per second.}$$

The speed of the belt is equal to the linear velocity of a point on the rim of the pulley. The radius of the pulley is $\frac{1}{2}$ foot, so that, by (2), the speed of the belt or pulley is

$$v = \tfrac{1}{2} \times 52.36 = 26.18 \text{ feet per second.}$$

EXERCISES

In the following exercises, r represents the radius of a circle, θ a central angle, and a the length of the subtended arc. In each case find the required number.

	r	θ	a	*Answer*
1.	10 in.	2 radians	required	20 in.
2.	3 ft.	8 radians	required	
3.	564.73 in.	$\pi/6$ radians	required	295.69 in.
4.	7046.9 ft.	$2\pi/3$ radians	required	
5.	40 ft.	required in radians	30 ft.	.75 radians
6.	21 ft.	required in radians	35 ft.	
7.	209.78 rods	required in radians	318.69 rods	1.5191 radians
8.	1243.6 ft.	required in radians	697.32 ft.	
9.	required	$\pi/2$ radians	600 ft.	382 ft.
10.	required	3 radians	627 ft.	
11.	1653.8 ft.	17°	required	490.7 ft.
12.	407.35 rods	46°	required	
13.	637.18 rods	required in degrees	214.79 rods	19° 18.8′
14.	427.6 ft.	required in degrees	516.5 ft.	
15.	required	38° 18.4′	776.4 ft.	1161.3 ft.
16.	required	29° 48′	1024.3 ft.	

17. A railroad curve is laid out on a circle of radius 723 feet. What is its length, if it subtends an angle of 23° 17′ at the center?

Ans. 293.8 ft.

18. A pendulum 37.46 inches long swings through an arc of 3.63 inches. Find the angle through which it swings, in degrees.

19. An automobile has tires 30.5 inches in diameter. (*a*) How many revolutions will a wheel make in one mile? (*b*) Through how many radians will the wheel turn in one mile?

Ans. (*a*) 661.2 revs. (*b*) 4154.7 rads.

20. A pulley 18.7 inches in diameter is driven by a belt. (*a*) How far will the belt travel while the pulley makes 100 revolutions? (*b*) How far will it travel while the pulley turns through 6 radians?

21. What is the angular velocity of a wheel which makes 20 revolutions per second? *Ans.* 40π rads. per sec., or 7200 deg. per sec.

22. How many revolutions per minute are made by a wheel which turns with an angular velocity of 12 radians per second?

23. What is the linear velocity of a pulley 20 inches in diameter making 500 revolutions per minute? *Ans.* 43.63 ft. per sec.

24. What is the radius of a pulley if a belt traveling at the rate of 56 feet per second drives it at the rate of 360 revolutions per minute?

25. A wheel 29 inches in diameter is turning with an angular velocity of 6.4 radians per second. What is the linear velocity of a point on the rim? *Ans.* 7.7 ft. per sec.

26. The rim of a wheel 42 inches in diameter has a linear velocity of 18 feet per second. What is the angular velocity in radians per second? In degrees per second?

27. What is the radius of a wheel which has an angular velocity of 264 degrees per second when driven by a chain on the wheel which travels 140 feet per minute? *Ans.* 6.08 ins.

28. The minute hand of a clock is 6 inches long. How far does its end travel in 25 minutes?

29. An automobile has tires 32 inches in diameter. When the car travels 20 miles an hour how fast does a wheel turn on its axle in revolutions per second? What is its angular velocity in radians per second? Take $\pi = 22/7$. *Ans.* $3\frac{1}{2}$ r.p.s.; 22 radians per sec.

30. An airplane propellor makes 1975 revolutions per minute, and is 7 feet 9 inches from tip to tip. What is the linear velocity of its tip?

CHAPTER VI

PROPERTIES OF THE TRIGONOMETRIC FUNCTIONS

49. Related angles. — Our tables of trigonometric functions are formed only for positive acute angles. If we are to use them to find the functions of other angles, we must therefore be able to express the functions of any angle in terms of the functions of an acute angle. This we may do by the use of a *related angle*. If θ is any angle, we shall call the related angle of θ the positive acute angle which, when added to or subtracted from θ, gives an angle coterminal with either 180° or 360°.

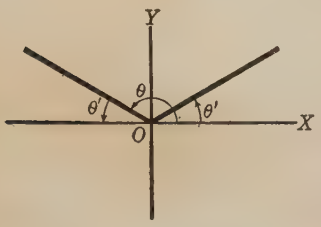

FIG. 39

EXAMPLES. In the figure, θ' is the related angle of θ. The related angle of 216° is 36°, since $216° - 36° = 180°$. The related angle of 336° is 24°, since $336° + 24° = 360°$.

EXERCISES

For each of the following angles find the related angle.

1. 98°	*Ans.* 82°.	**2.** 164°.
3. 200°.	*Ans.* 20°.	**4.** 252°.
5. 295°.	*Ans.* 65°.	**6.** 340°.
7. −476°.	*Ans.* 64°.	**8.** 848°.
9. −112°.	*Ans.* 68°.	**10.** −17°.

50. A standard construction. — In the following paragraphs, when any two angles θ and θ' are considered we shall suppose that (1) θ and θ' are in standard position on a set of coördinate axes; (2) P is some

conveniently chosen point on the terminal side of θ; (3) (x, y) and r are the coördinates and radius vector of P; (4) P' is the point on the terminal side of θ' which is *at the same distance, $r' = r$, from the origin* as is P; (5) (x', y') are the coördinates of P'; (6) M and M', respectively, are the feet of the perpendiculars drawn from P and P' to the x-axis.

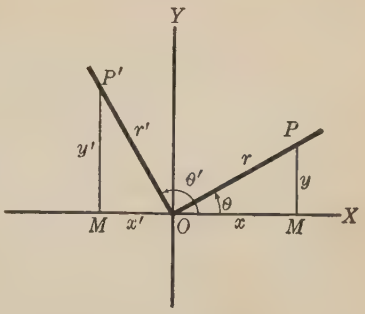

FIG. 40

EXERCISES

Draw the figure for this standard construction for each of the following cases:

1. $\theta = 60°$, $\theta' = 150°$.
2. $\theta = 30°$, $\theta' = -30°$.
3. $\theta = 210°$, $\theta' = -210°$.
4. $\theta = 150°$, $\theta' = \theta + 90°$.
5. $\theta = 240°$, $\theta' = \theta - 180°$.
6. $\theta = 240°$, $\theta' = \theta - 270°$.

In each of the following cases show that the triangles OMP and $OM'P'$ are equal, when θ is an acute angle.

7. $\theta' = -\theta$.
8. $\theta' = 180° - \theta$.
9. $\theta' = \theta + 180°$.
10. $\theta' = 360° - \theta$.

In the following cases show that triangles OMP and $OM'P'$ are equal for all values of θ. Draw the figure for at least two values of θ taken in different quadrants.

11. $\theta' = -\theta$.
12. $\theta' = \theta + 90°$.
13. $\theta' = \theta + 180°$.
14. $\theta' = \theta + 270°$.
15. θ' = the related angle of θ.

51. Functions of related angles. — *Theorem: Any trigonometric function of an angle is numerically equal to the same function of the related angle.* To prove this, let θ be any angle (shown in second, third and fourth quadrants in Figure 41). Denote by θ' its related

angle, and make the standard construction of Article **50**. The right triangles OMP and $OM'P'$ are then equal (Example 15, Article **50**), and their parts correspond so that r and r', x and x', y and y' are

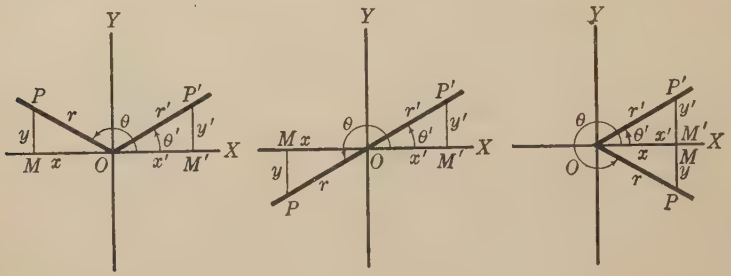

Fig. 41

numerically equal (that is, equal except perhaps for sign). Consequently

$$\frac{y}{r} \text{ and } \frac{y'}{r'}, \qquad \frac{x}{r} \text{ and } \frac{x'}{r'}, \qquad \frac{y}{x} \text{ and } \frac{y'}{x'},$$
$$\frac{x}{y} \text{ and } \frac{x'}{y'}, \qquad \frac{r}{x} \text{ and } \frac{r'}{x'}, \qquad \frac{r}{y} \text{ and } \frac{r'}{y'}$$

are numerically equal. Therefore

$\sin\theta$ and $\sin\theta'$, $\cos\theta$ and $\cos\theta'$, $\tan\theta$ and $\tan\theta'$,
$\cot\theta$ and $\cot\theta'$, $\sec\theta$ and $\sec\theta'$, $\csc\theta$ and $\csc\theta'$

are numerically equal. Any trigonometric function of an angle is numerically equal to the same function of the related angle. The sign of a given function of θ can be determined by the quadrant in which θ terminates.

If θ is an angle between 90° and 180°, we can write it as $(180° - \alpha)$ where α is the related angle of θ. If θ is between 180° and 270°, $\theta = 180° + \alpha$; and if θ is between 270° and 360°, $\theta = 360° - \alpha$, where, in each case, α is the related angle of θ. We can therefore say that *if α is any acute angle, any function of the angle* $(180° - \alpha)$

or of the angle $(180° + \alpha)$ *or of the angle* $(360° - \alpha)$ *is numerically equal to the same function of* α. Later on in this chapter it will be shown that this statement is true when α is *any* angle.

EXAMPLE 1. Express the sine and cotangent of 147° 35′ as functions of an acute angle. The related angle is 32° 25′. Since 147° 35′ is in the second quadrant, its sine is positive and its cotangent is negative. Therefore

$$\sin 147° 35' = \sin 32° 25',$$
$$\cot 147° 35' = -\cot 32° 25'.$$

EXAMPLE 2. Find the sine and cosine of 163° 12′. The related angle is 16° 48′. The desired numbers are numerically equal to sin 16° 48′ = .28903 and cos 16° 48′ = .95732. Since 163° 12′ is in the second quadrant, its sine is positive and its cosine is negative. Therefore sin 163° 12′ = .28903, and cos 163° 12′ = −.95732.

EXAMPLE 3. Find the sine and tangent of 258° 23′. The related angle is 78° 23′. Since 258° 23′ is in the third quadrant, its sine is negative and its tangent is positive. Therefore

$$\sin 258° 23' = -\sin 78° 23' = -.97952,$$
$$\tan 258° 23' = \tan 78° 23' = 4.8644.$$

EXAMPLE 4. By logarithms find the value of

$$x = 47.61 \times \cos 212° 16'.$$

We have

$$x = 47.61 \times (-\cos 32° 16') = -47.61 \times \cos 32° 16'.$$

Negative numbers have no real logarithms. We shall therefore first compute the value of

$$47.61 \times \cos 32° 16',$$

and then prefix the negative sign.

$$\log 47.61 = 1.67770$$
$$\log \cos 32° 16' = 9.92715 - 10$$
$$1.60485$$

$$-x = 40.26, \qquad x = -40.26.$$

EXERCISES

Without using tables, find the values of the trigonometric functions of the following angles.

1. 120°.	**2.** 135°.	**3.** −30°.
4. −45°.	**5.** 210°.	**6.** 240°.
7. −150°.	**8.** −120°.	**9.** 225°.
10. 315°.	**11.** 300°.	**12.** −180°.
13. 480°.	**14.** 660°.	**15.** 570°.
16. 780°.	**17.** −405°.	**18.** −495°.

Using the five-place tables, find the values of the following functions.

19. sin 114° 26′. *Ans.* .91044.	**20.** cos 172° 51′.
21. tan 197° 14′. *Ans.* .31019.	**22.** cot 256° 18′.
23. cos 538° 46′. *Ans.* −.99977.	**24.** sin 324° 19′.
25. cot 462° 12′. *Ans.* −.21621.	**26.** tan 283° 11′.
27. sin 211° 53′. *Ans.* −.52819.	**28.** tan 654° 17′.
29. cos (−100° 12′). *Ans.* −.17708.	**30.** cot (−267° 54′).
31. cot 131° 16.9′. *Ans.* −.87796.	**32.** sin (−124° 28.3′).
33. tan 169° 13.7′. *Ans.* −.19025.	**34.** cos 101° 15.6′.

Find the values of the following numbers by logarithms.

35. 28.36 × sin 124° 42′.	*Ans.* 23.32.
36. 11.32 × cos 301° 12′.	
37. 304.7 × tan 227° 33′.	*Ans.* 333.1.
38. 294.6 × cot 254° 48′.	
39. 103.6 × cos 147° 16′.	*Ans.* −87.15.
40. 7203 × tan 129° 29′.	
41. 6.048 × cot 298° 24′.	*Ans.* −3.270.
42. .05679 × sin 342° 24′.	
43. 17.632 × cos 212° 13.4′.	*Ans.* −14.916.
44. 399.536 ÷ cos 186° 42.3′.	
45. 41.763 × csc 304° 17′.	*Ans.* −50.544.
46. 501.43 × sec 501° 43′.	

52. Functions of $(-\theta)$. — Let θ be any angle, and let $\theta' = -\theta$. Make the standard construction of Article **50**. The triangles OMP and $OM'P'$ are equal (Example 11, Article **50**). Therefore points P and P' are symmetrically placed on opposite sides of the x-axis, and $x' = x$ and $y' = -y$. Consequently

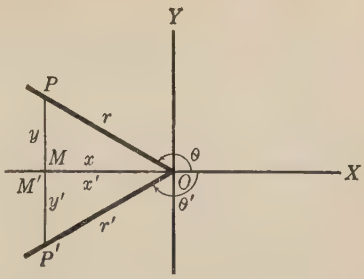

FIG. 42

$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\sin\theta, \qquad \cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta, \qquad \cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\cot\theta,$$

$$\sec(-\theta) = \frac{r'}{x'} = \frac{r}{x} = \sec\theta, \qquad \csc(-\theta) = \frac{r'}{y'} = \frac{r}{-y} = -\csc\theta.$$

These results can be expressed as follows: *Changing the sign of an angle without changing its numerical value, leaves the cosine and secant of the angle unchanged, but reverses the signs of all the other functions without changing their numerical values.*

We sometimes say that an expression which is not changed in any way when the sign of the variable is reversed is an *even function* of the variable. Thus any *even* power of x is an even function of x, since x^2, x^4, x^6, etc., are unchanged if x is replaced by $(-x)$. Hence the name even function. Likewise, a function such as x, x^3, x^5 or any odd power of x, which reverses its sign but is unchanged in numerical value when the sign of the variable is changed, is called an *odd function* of the variable. We can say, then, that $\cos\theta$ and $\sec\theta$ are even functions of θ, and all the other trigonometric functions are odd functions of θ.

If θ is any angle, $360° - \theta$ is coterminal with $(-\theta)$. The functions of $(360° - \theta)$ are therefore numerically equal to the same functions of θ.

★**53. Functions of $(\theta + 90°)$.** — Let θ be any angle, and let $\theta' = \theta + 90°$. Make the standard construction of Article **50**.

The angle θ may be in any quadrant and the figure is drawn for each quadrant. The right triangles OMP and $OM'P'$ are equal (Example

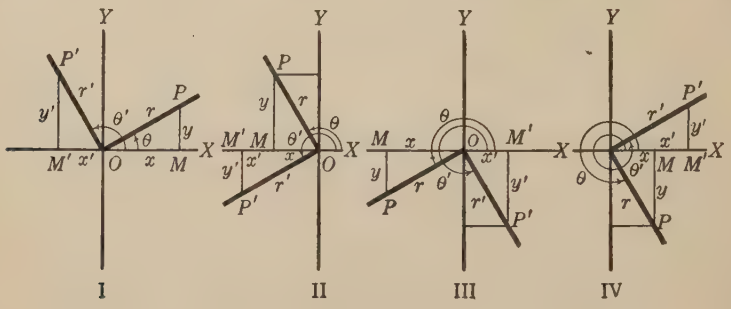

Fig. 43

12, Article 50), the horizontal side of one being equal to the vertical side of the other. Taking account of the signs, we have, *for all four figures:*

$$x' = -y, \qquad y' = x, \qquad r' = r.$$

Therefore, by the definitions of the trigonometric functions,

$$\sin\theta' = \frac{y'}{r'} = \frac{x}{r} = \cos\theta, \qquad \cos\theta' = \frac{x'}{r'} = \frac{-y}{r} = -\sin\theta,$$

$$\tan\theta' = \frac{y'}{x'} = \frac{x}{-y} = -\cot\theta, \qquad \cot\theta' = \frac{x'}{y'} = \frac{-y}{x} = -\tan\theta,$$

$$\sec\theta' = \frac{r'}{x'} = \frac{r}{-y} = -\csc\theta, \qquad \csc\theta' = \frac{r'}{y'} = \frac{r}{x} = \sec\theta,$$

or

$$\sin(\theta + 90°) = \cos\theta, \qquad \cos(\theta + 90°) = -\sin\theta,$$
$$\tan(\theta + 90°) = -\cot\theta, \qquad \cot(\theta + 90°) = -\tan\theta,$$
$$\sec(\theta + 90°) = -\csc\theta, \qquad \csc(\theta + 90°) = \sec\theta.$$

These and similar formulas should always be translated by the student into words, as, for example:

If two angles differ by 90°, the sine of the larger angle is equal to the cosine of the smaller angle; the cosine of the larger is equal to the negative of the sine of the smaller, etc.

★**54. Functions of (θ − 90°).** — If, in the results of the preceding paragraph, we indicate the *larger* angle by θ, the smaller one will be $(\theta - 90°)$, and we have

$$\begin{aligned} \sin(\theta - 90°) &= -\cos\theta, & \cos(\theta - 90°) &= \sin\theta, \\ \tan(\theta - 90°) &= -\cot\theta, & \cot(\theta - 90°) &= -\tan\theta, \\ \sec(\theta - 90°) &= \csc\theta, & \csc(\theta - 90°) &= -\sec\theta. \end{aligned}$$

Combining these results with those for the functions of $(-\theta)$, since $(90° - \theta)$ is the negative of $(\theta - 90°)$, we have

$$\begin{aligned} \sin(90° - \theta) &= \cos\theta, & \cos(90° - \theta) &= \sin\theta, \\ \tan(90° - \theta) &= \cot\theta, & \cot(90° - \theta) &= \tan\theta, \\ \sec(90° - \theta) &= \csc\theta, & \csc(90° - \theta) &= \sec\theta. \end{aligned}$$

★**55. Functions of (θ ± 180°).** — The functions of $(\theta + 180°)$ can be expressed in terms of the functions of θ by adding 90° to $(\theta + 90°)$ and applying the results of Article **53** twice. Thus, we may write

$$\sin(\theta + 180°) = \sin[(\theta + 90°) + 90°] = \cos(\theta + 90°) = -\sin\theta.$$

We obtain the formulas

$$\begin{aligned} \sin(\theta + 180°) &= -\sin\theta, & \cos(\theta + 180°) &= -\cos\theta, \\ \tan(\theta + 180°) &= \tan\theta, & \cot(\theta + 180°) &= \cot\theta, \\ \sec(\theta + 180°) &= -\sec\theta, & \csc(\theta + 180°) &= -\csc\theta. \end{aligned}$$

The functions of $(\theta - 180°)$ are the same as those of $(\theta + 180°)$ since these two angles have the same terminal sides. Since $(180° - \theta)$ is the negative of $(\theta - 180°)$, its functions are numerically the same as the same functions of θ.

EXERCISE: Write out formulas for $\sin(180° - \theta)$, $\cos(180° - \theta)$, etc., in terms of the functions of θ.

★**56. Functions of ($\theta \pm 270°$).** — The functions of $(\theta + 270°)$ may be expressed in terms of the functions of θ by combining the results of Articles **53** and **55**. Thus,

$$\sin(\theta + 270°) = \sin[(\theta + 180°) + 90°] = \cos(\theta + 180°) = -\cos\theta.$$

We can thus obtain definite formulas which hold for all values of θ. It is best to remember, however, merely the fact that any function of $(\theta \pm 270°)$ is numerically the same as the cofunction of θ, the algebraic sign being determined by the quadrant in which the angle terminates.

Exercise: Write formulas for the functions of $(\theta + 270°)$, $(\theta - 270°)$ and $(270° - \theta)$ in terms of the functions of θ.

★**57. Summary.** — Any function of $(-\theta)$, $(\theta + 180°)$, $(\theta - 180°)$, $(180° - \theta)$, or of $(360° - \theta)$ is numerically equal to the same function of θ. Any function of $(\theta + 90°)$, $(\theta + 270°)$, $(90° - \theta)$, $(\theta - 270°)$, or $(270° - \theta)$ is numerically equal to the cofunction of θ. The algebraic sign to be used in the formula connecting a function of any one of these angles with the same function or cofunction of θ will be the same whether θ is in the first or in some other quadrant. To determine the algebraic sign, then, assume that θ is in the first quadrant, and construct the given angle. From the figure determine whether the required function is plus or minus the function or cofunction of θ.

We need not consider especially such angles as $\theta + 630°$, since the functions of this angle are the same as the functions of the coterminal angle $(\theta + 630°) - 360° = \theta + 270°$, which has been considered above.

Example 1. Express $\cos(540° + \theta)$ as a function of θ. We write $\cos(540° + \theta) = \cos(180° + \theta)$. This is *numerically* equal to $\cos\theta$. To determine whether it has the same sign as $\cos\theta$ or the opposite sign, we see that *if* θ is in the first quadrant (so that $\cos\theta$ is positive), $(\theta + 180°)$ is in the third quadrant, so that its cosine is negative. Therefore

$$\cos(\theta + 540°) = -\cos\theta,$$

for all values of θ.

EXAMPLE 2. Express $\tan(7\pi/2 - \theta)$ as a function of θ. We have

$$\tan\left(\frac{7\pi}{2} - \theta\right) = \tan\left[\left(\frac{7\pi}{2} - \theta\right) - 2\pi\right] = \tan\left(\frac{3\pi}{2} - \theta\right).$$

Numerically this is equal to $\cot\theta$, since $3\pi/2$ radians equals 270°. If θ is in the first quadrant, $(3\pi/2 - \theta)$ is in the third quadrant. Therefore its tangent is positive, and

$$\tan(7\pi/2 - \theta) = \cot\theta$$

for all values of θ.

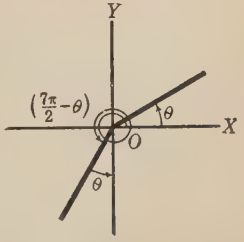

FIG. 44

EXERCISES

Express each of the following functions as a function of θ.

1.	$\sin(270° + \theta)$.	*Ans.*	$-\cos\theta$.	**2.**	$\cos(\theta - 270°)$.
3.	$\cos(540° + \theta)$.	*Ans.*	$-\cos\theta$.	**4.**	$\tan(\theta + 180°)$.
5.	$\tan(810° + \theta)$.	*Ans.*	$-\cot\theta$.	**6.**	$\cot(450° + \theta)$.
7.	$\cot(\theta - 630°)$.	*Ans.*	$-\tan\theta$.	**8.**	$\sin(1080° + \theta)$.
9.	$\tan(540° + \theta)$.	*Ans.*	$\tan\theta$.	**10.**	$\cos(720° - \theta)$.
11.	$\sin(810° - \theta)$.	*Ans.*	$\cos\theta$.	**12.**	$\tan(900° - \theta)$.
13.	$\csc(900° - \theta)$.	*Ans.*	$\csc\theta$.	**14.**	$\sin(630° - \theta)$.
15.	$\cos(\pi + \theta)$.	*Ans.*	$-\cos\theta$.	**16.**	$\tan(3\pi/2 + \theta)$.
17.	$\cot(4\pi - \theta)$.	*Ans.*	$-\cot\theta$.	**18.**	$\sec(5\pi/2 + \theta)$.
19.	$\csc(9\pi/2 + \theta)$.	*Ans.*	$\sec\theta$.	**20.**	$\tan(7\pi/2 - \theta)$.

58. Periodic functions. — We have seen that if an angle is increased by 360° or 2π radians, the value of the sine of the angle is unchanged. That is, if θ is any angle,

$$\sin(\theta + 2\pi) = \sin\theta.$$

The values of the sine of an angle thus recur *periodically*, the period of recurrence being 2π.

An expression in θ which, for every value of θ, is unchanged when θ is increased by p, is said to be a *periodic function of θ with period p*. The values assumed by the function in any interval of length p will be taken on in any other interval of the same length.

The trigonometric functions are all periodic functions of period 2π radians or 360°. This is the smallest or fundamental period for the sine, cosine, secant and cosecant. But the tangent and cotangent have also a smaller period equal to π radians or 180°. For, by Article **55**, $\tan(\theta + 180°) = \tan\theta$, for all values of θ.

59. The functions of a variable angle. — We shall now study the changes that are brought about in the values of the trigonometric functions of an angle by changes in the value of the angle itself. Beginning with the angle zero, we shall let the angle increase gradually. Its functions will then vary in value, each function in its own way. We need to let the angle vary only from 0° to 360°, for other angles lead to no new positions of the terminal side (the angle being taken in standard position) and therefore to no new values of the trigonometric functions.

60. The variation of the sine and cosine. — Let θ be any angle, in standard position. Then $\sin\theta = y/r$ and $\cos\theta = x/r$, where x, y, and r are the coördinates and radius vector of any point P on the terminal side of θ. For convenience take P so that r is always equal to 1. Then, as θ varies, the point P moves along the circle drawn about the origin as center with one unit as radius.

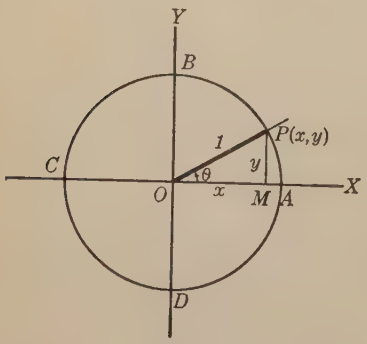

FIG. 45

Since $r = 1$,

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y,$$

and

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x.$$

That is, $\sin\theta$ is a number equal to the number of units in the length of y, due regard being had to sign. For example, when θ has a certain value equal to about 37°, point P on the circle has coördinates (4/5, 3/5). Since the length of y is 3/5 of a unit, $\sin\theta = 3/5$. Similarly, since $x = 4/5$, $\cos\theta = 4/5$.

If θ starts at 0° and increases to 90°, P starts at A, and moves along the circle to B, so that $\sin\theta$ (or y) starts at 0 and increases to 1. And $\cos\theta$ (or x) starts at 1, and decreases to 0. In fact:

When θ increases	*P moves*	*$\sin\theta$ (or y)*	*$\cos\theta$ (or x)*
from 0° to 90°	from A to B	increases from 0 to 1,	decreases from 1 to 0;
from 90° to 180°	from B to C	decreases from 1 to 0,	decreases from 0 to −1;
from 180° to 270°	from C to D	decreases from 0 to −1,	increases from −1 to 0;
from 270° to 360°	from D to A	increases from −1 to 0,	increases from 0 to 1.

61. The variation of the tangent and cotangent. — In studying the function $\tan\theta = y/x$, we keep x numerically equal to 1. For angles in the first and fourth quadrants, P then lies on the line AP (the tangent at A to the unit circle, whence the name "tangent"). Therefore

$$\tan\theta = \frac{y}{x} = \frac{y}{1} = y.$$

If θ is in the second or third quadrant, P lies on BP', and

$$\tan\theta = \frac{y}{x} = \frac{y}{-1} = -y.$$

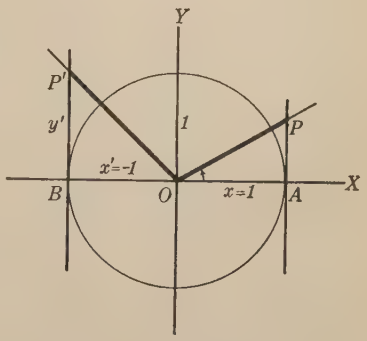

FIG. 46

When $\theta = 0°$, P is at A, $y = 0$, and therefore $\tan 0° = 0$. As θ increases, P moves up the line AP, and y increases. Therefore $\tan\theta$ increases, and continues to increase so long as θ is increasing while still less than 90°. As θ draws near to 90°, P moves very far up on the line AP, and y or $\tan x$ increases without limit. If θ is exactly 90°, the point P will be somewhere on the y-axis, and, whatever the value of y may be, the value of x is zero. Therefore, since y/x or $y/0$ has no meaning because division by zero is impossible, the angle 90° has no tangent. Or, as we say, *the tangent of 90° is not defined.*

However, when θ is nearly 90°, the point P is very far up on the line AP, and $y = \tan\theta$ is very large. In fact, by taking θ less than 90° but near enough to 90°, we can evidently take P as high as we wish, that is, make $\tan\theta$ as great as we desire. We may write $\tan 90° = \infty$, which may be read "the tangent of 90° is infinite," but it must be remembered that *this is merely an abbreviation for the statement:* "90° has no tangent; by taking θ less than 90° but sufficiently close to 90° we can make $\tan\theta$ as large as we wish." This is a statement that the student should remember verbatim.

For angles in the second quadrant, $x = -1$, and $\tan\theta = -y$. As θ increases from 90° to 180°, y decreases to 0 after passing through all positive values, and therefore $\tan\theta$ is negative and increases through all negative values to 0.

The student should study for himself the variation in $\tan\theta$ as θ increases from 180° to 360° and compare the results with the table of Article **62.**

Cot θ. The variation of $\cot\theta$ should be traced from the fact that $\cot\theta = 1/\tan\theta$. Since $\tan\theta$ increases continuously, $\cot\theta$ decreases. When $\tan\theta$ becomes infinite, $\cot\theta$ becomes zero, and when $\tan\theta$ approaches zero, $\cot\theta$ becomes infinite.

62. The variation of the secant and cosecant. — *Sec* θ. We think of $\sec\theta$ as the reciprocal of $\cos\theta$; $\sec\theta = 1/\cos\theta$. When either of these functions increases the other decreases.

If θ increases from 0° to 90°, $\cos\theta$ starts at 1 and decreases. Therefore $\sec\theta$ starts at 1 and increases. Now $\cos 90° = 0$, and therefore $\sec 90°$ is not defined. By taking θ sufficiently close to 90° we can make $\sec\theta$ as large as we wish. Using the notation in the same sense as for the tangent, we can write $\sec 90° = \infty$. *The student should repeat the entire phrase of which this is an abbreviation.*

When θ varies from 90° to 180°, $\cos\theta$ decreases from 0 to -1, and $\sec\theta$ increases up to -1 after passing through values which are negative but numerically very large. The student should trace the variation of $\sec\theta$ as θ increases from 180° to 360° and compare the results with the table on page 93.

Csc θ. It is left to the student to trace the variation of csc θ, recalling that $\csc\theta = 1/\sin\theta$.

The foregoing results may be summarized as follows:

Quadrant	I		II		III		IV	
Variation	*From*	*To*	*From*	*To*	*From*	*To*	*From*	*To*
θ	0	90°	90°	180°	180°	270°	270°	360°
$\sin\theta$	0	1	1	0	0	−1	−1	0
$\cos\theta$	1	0	0	−1	−1	0	0	1
$\tan\theta$	0	∞	$-\infty$	0	0	∞	$-\infty$	0
$\cot\theta$	∞	0	0	$-\infty$	∞	0	0	$-\infty$
$\sec\theta$	1	∞	$-\infty$	−1	−1	$-\infty$	∞	1
$\csc\theta$	∞	1	1	∞	$-\infty$	−1	−1	$-\infty$

We should notice particularly that the sine and cosine never exceed 1 in numerical value; the tangent and cotangent may assume any value; the secant and cosecant take on no values numerically less than 1. *In the first quadrant* the sine, tangent, and secant increase when the angle increases, but their cofunctions, the cosine, cotangent, and cosecant decrease.

EXERCISES

Using Figure 3, make a table of the values of the sine, cosine, tangent, and cotangent for multiples of 20° from 0° to 360°.

63. The graph of $y = \sin x$. — In the following discussion, x will be used to denote the value of an angle and y to denote the value of the sine of that angle. They are no longer the coördinates of a point on the terminal side of some angle. Let us draw a set of coördinate axes. Distances on the x-scale now measure angles, and distances on the y-scale will represent values of the sine. Corresponding to any value of the angle x we can determine (from a table or otherwise) the value of $y = \sin x$, and plot the point whose coördinates are these values of x and y. The whole set of points that can be obtained in

this way form a smooth curve which is the graph of the function $y = \sin x$.

The choice of scale, to a certain extent, is arbitrary. But *in graphing the trigonometric functions we shall always assume that x is expressed in radian measure,* so that *the same distance which represents the number* 1 *on the vertical scale represents one radian on the horizontal scale.*

In applications of the graphs, particularly in the calculus, the use of any other scale leads to very awkward forms and complicated formulas. Since π is approximately equal to 3.14, the vertical distance representing the number 1 is slightly less than one-third of the horizontal distance representing π radians.

We can predict the general appearance of the curve from the results summarized in Article **62**. From any table of sines we can obtain as many points as we wish for values of x from $x = 0$ to $x = 2\pi$. Since $y = \sin x$ is periodic with period 2π, the curve obtained for the interval from 0 to 2π will be reproduced in the intervals 2π to 4π, -2π to 0, 4π to 6π, etc., and may be continued as far as we like to the right or left. Figure 47 is obtained by plotting the points from the following table.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \sin x$	0	.50	.87	1	.87	.50	0	$-.50$	$-.87$	-1	$-.87$	$-.50$	0

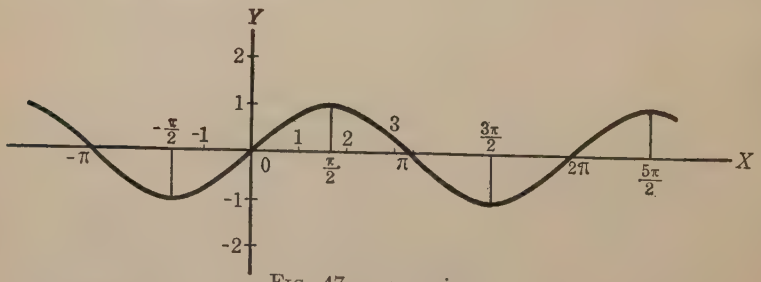

FIG. 47 — $y = \sin x$.

The student should learn to draw a hasty sketch of this graph from memory.

★**64. Another construction.** — The graph of $y = \sin x$ may be drawn in the following way without using a table. Take any convenient point, C, on the x-axis. With C as center, draw a circle with radius one unit. Let x be any number, and with its vertex at C lay off from the x-axis the angle ACP equal to x radians. The arc AP which it subtends is in length equal to x units (Article **47**), and $\sin x$ is equal to the number of units in the height MP (Article **60**). Lay

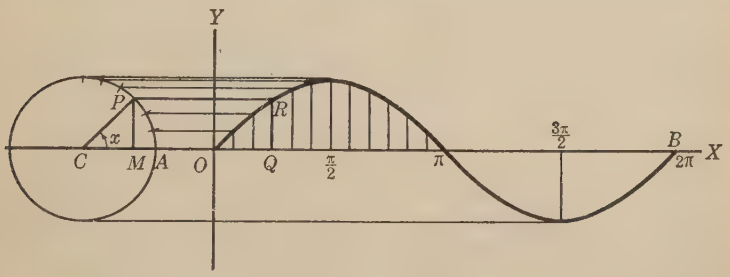

FIG. 48 — $y = \sin x$.

off the distance OQ equal to the length of the arc AP (which is equal to x). Vertically through Q draw QR equal to MP (which is equal to $\sin x$). The point R is therefore in the graph of $y = \sin x$, since its abscissa is x and its ordinate is y. We can find as many such points as we wish. In order to find points P and Q that correspond, so that OQ equals arc AP we can lay off the distance OB equal to 2π (which equals the circumference of our circle), and then divide the circumference and OB into some convenient number of equal parts. P and Q will then be corresponding points of division.

EXERCISES

1. Draw the graph of $y = \cos x$ by one of the above methods, and also by using the graph of $y = \sin x$ and the fact that $\cos x = \sin (x + \pi/2)$.

★2. An engine piston is connected by a rod to a flywheel which turns with uniform speed. Show how the position of the piston can be represented by means of a trigonometric function.

65. The graph of $y = \tan x$. — The most characteristic features of the graph are: (1) the curve always slopes upward toward the right; (2) the function becomes positively infinite as x tends toward any odd multiple of $\pi/2$ from the left, and negatively infinite as x tends toward such a value from the right; (3) the graph crosses the x-axis when x is any multiple of π; (4) the function has the period π, so that there are two complete patterns of the function in any interval of length 2π. A very few plotted points will now suffice to determine the figure. The student should learn to sketch the figure quickly from memory.

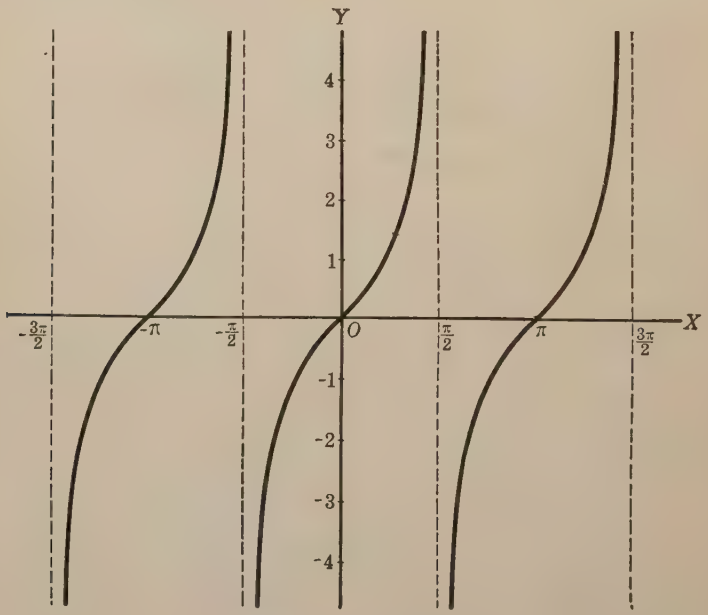

FIG. 49 — $y = \tan x$.

EXERCISE

Graph the function $y = \cot x$. Use the fact that this function is the reciprocal of $\tan x$, so that when one function becomes infinite the other becomes zero, and the graph of $y = \cot x$ slopes downward to the right.

★66. The graph of $y = \csc x$. — The graph of this function can be obtained from that of $y = \sin x$ of which it is the reciprocal. When $\sin x = \pm 1$, $\csc x$ has the same value. When x is equal to any multiple of π, $\sin x = 0$, and $\csc x$ becomes infinite.

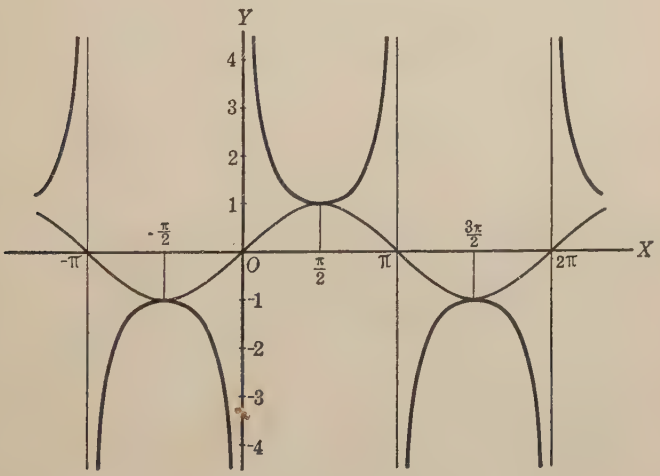

FIG. 50 — $y = \csc x$.

EXERCISE

Graph the function $y = \sec x$.

★67. The graphs of certain related functions. —

EXAMPLE 1. Graph the function $y = 3 \sin 2x$. This graph can be drawn very easily by considering the graph of $y = \sin x$. While x varies from 0 to π, $2x$

varies from 0 to 2π. Therefore in the interval from $x = 0$ to $x = \pi$, $\sin 2x$ goes through the same set of values that $\sin x$ takes on in the entire interval from $x = 0$ to $x = 2\pi$. And, $y = 3 \sin 2x$ varies between the limits 3 and -3. The graphs of $y = \sin x$ and $y = 3 \sin 2x$ are shown together in Figure 51.

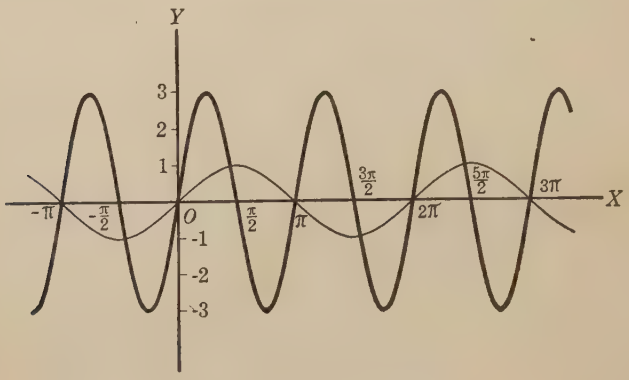

Fig. 51

Example 2. Graph the function $y = \sin x + \sin 2x$. This graph can be made most easily by the method known as composition of ordinates. We first draw, on the same axes, the graphs of the two separate terms $y = \sin x$ and $y = \sin 2x$. Then, the height of the graph of $y = \sin x + \sin 2x$ can be found

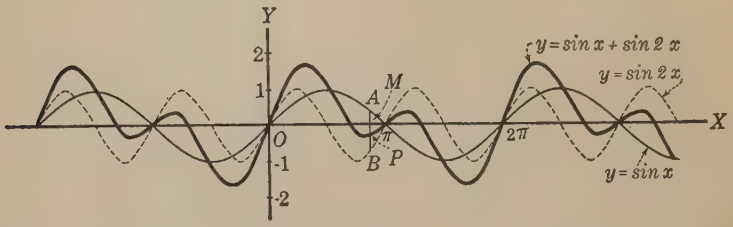

Fig. 52

for any value of x by adding the heights of the other two curves. Of course we must take the sign into account. Thus, to get the point P of Figure 52 we must measure the distance MB downward from the point A, since MB, which represents the value of $\sin 2x$, is negative at this point.

EXAMPLE 3. Graph the function $y = x + \sin x$. Graph the two functions $y = x$, which is represented by a straight line, and $y = \sin x$. The desired graph is then obtained as in the preceding example, by the composition of ordinates.

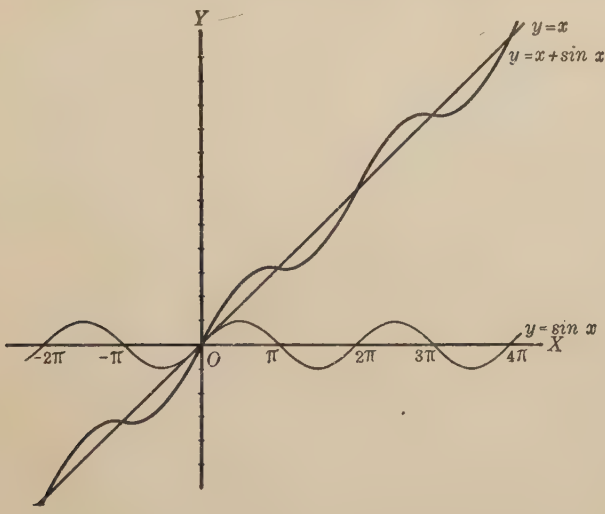

FIG. 53

Note: It may be objected here that x stands for a number in the first term of the function, and for an angle in the second term. As a matter of fact, however, *x represents a number in each term.* When we graph $y = \sin x$, we really assume that y is a function of the *number x. By the sine of the number x, we mean the sine of the angle the number of radians in which is equal to x.* Similar definitions hold for the other trigonometric functions of a number.

EXERCISES

Graph the following functions in the interval from $x = -2\pi$ to $x = 2\pi$.

1. $y = \sin 3x$.
2. $y = 2 \sin 4x$.
3. $y = \cos 2x$.
4. $y = -\cos 3x$.
5. $y = \csc 2x$.
6. $y = \sec 2x$.
7. $y = \sin x - \sin 2x$.
8. $y = \sin x + \cos x$.

9. $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$.

10. $y = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x$.

11. $y = \cos x + \cos 2x$.

12. $y = \cos x + \frac{1}{3}\cos 3x$.

13. $y = \tan x + \cot x$.

14. $y = x - \sin x$.

15. $y = 2x + \sin 2x$.

16. $y = \tan x + \sin x$.

CHAPTER VII

THE FUNDAMENTAL RELATIONS

68. The fundamental relations. — Using the notation of Article 8, if θ is any angle,

$$\text{(a) } \sin\theta = \frac{y}{r}, \qquad \text{(b) } \cos\theta = \frac{x}{r}, \qquad \text{(c) } \tan\theta = \frac{y}{x},$$

$$\text{(d) } \cot\theta = \frac{x}{y}, \qquad \text{(e) } \sec\theta = \frac{r}{x}, \qquad \text{(f) } \csc\theta = \frac{r}{y}.$$

Moreover, since x, y, and r are numerically equal to the sides of a right triangle, we have

$$\text{(g)} \quad x^2 + y^2 = r^2.$$

From these seven equations we obtain eight *fundamental relations* or *fundamental identities* that always connect the functions of an angle. Three of these, which we may call the *reciprocal relations*, we have already proved and used repeatedly. They are:

$$\text{(1)} \qquad \csc\theta = \frac{1}{\sin\theta},$$

$$\text{(2)} \qquad \sec\theta = \frac{1}{\cos\theta},$$

$$\text{(3)} \qquad \cot\theta = \frac{1}{\tan\theta}.$$

To prove the first of these, for example, by equations (a) and (b) we have

$$\csc\theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin\theta}.$$

Two others, the *ratio relations*, are obtained with almost equal ease:

$$\frac{\sin\theta}{\cos\theta} = \tan\theta, \tag{4}$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta. \tag{5}$$

To get (4), we use equations (a), (b), (c), writing

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta.$$

The remaining three fundamental relations, or *squares relations*, are:

$$\sin^2\theta + \cos^2\theta = 1, \tag{6}$$

$$\tan^2\theta + 1 = \sec^2\theta, \tag{7}$$

$$\cot^2\theta + 1 = \csc^2\theta. \tag{8}$$

They are obtained by dividing equation (g) by r^2 to get (6), by x^2 to get (7), and by y^2 to get (8). Thus, on dividing equation (g) by r^2 we have

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

which, by (a) and (b), is

$$\cos^2\theta + \sin^2\theta = 1.$$

The student should memorize the fundamental identities at once, and learn, also, to recognize them in modified forms. The relation (6), for example, is often useful in one of the forms

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta} \qquad \text{or} \qquad \cos\theta = \pm\sqrt{1 - \sin^2\theta},$$

for, in such a form, it gives us the value of the sine or the cosine when the other is known.

Example 1. Given that x is an angle in the second quadrant, and that $\sin x = 5/13$, find the other functions of x. From (6) we have

$$\cos x = \pm\sqrt{1 - \sin^2 x} = \pm\sqrt{1 - \tfrac{25}{169}} = \pm\tfrac{12}{13}.$$

However, since x is in the second quadrant, the cosine is negative, and $\cos x = -12/13$. From (4),

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}.$$

The relations (1), (2), and (3) then give

$$\csc x = 13/5, \qquad \sec x = -13/12, \qquad \cot x = -12/5.$$

This is an *algebraic* method of finding the values of the other functions when one of them is known. In an earlier chapter (Article **9**) we solved the same problem geometrically.

Example 2. Given that ϕ is an angle in the fourth quadrant and that $\tan \phi = -3$, find the other functions of ϕ. By relation (7) we have

$$\sec \phi = \sqrt{1 + \tan^2 \phi} = \sqrt{1 + 9} = \sqrt{10}.$$

Then, by (2), $\cos \phi = \dfrac{1}{\sqrt{10}} = \frac{1}{10}\sqrt{10}$; and by (3), $\cot \phi = -\frac{1}{3}$.

To find $\sin \phi$, we use (4), taking it in the form

$$\sin \phi = \tan \phi \cdot \cos \phi = -3 \cdot \tfrac{1}{10}\sqrt{10} = -\tfrac{3}{10}\sqrt{10},$$

and

$$\csc \phi = \frac{1}{\sin \phi} = -\tfrac{1}{3}\sqrt{10}.$$

EXERCISES

1. Prove the relations (5), (7), and (8).
2. State each of the fundamental relations in words.

From equations (1) to (8), obtain the following relations.

3. $\sin x \csc x = 1$.
4. $\tan x \cot x = 1$.
5. $\sin x = \pm\sqrt{1 - \cos^2 x}$.
6. $\csc x = \pm\sqrt{\cot^2 x + 1}$.
7. $\cot x = \pm\sqrt{\csc^2 x - 1}$.
8. $\tan x = \dfrac{\sec x}{\csc x}$.

9. $\cos x = \dfrac{\sin x}{\tan x}$.

10. $\cos x = \sin x \cot x$.

11. $\sin x = \dfrac{\tan x}{\pm\sqrt{1 + \tan^2 x}}$.

12. $\cos x = \dfrac{\cot x}{\pm\sqrt{1 + \cot^2 x}}$.

In the following examples use the fundamental relations to obtain all the functions not given.

13. $\sin \alpha = -3/5$, α in third quadrant.

Ans. $\cos \alpha = -4/5$, $\tan \alpha = 3/4$, $\cot \alpha = 4/3$, $\sec \alpha = -5/4$, $\csc \alpha = -5/3$.

14. $\cos \alpha = 5/6$, α in fourth quadrant.

15. $\tan x = -5/7$, x in fourth quadrant.

Ans. $\sin x = -\frac{5}{74}\sqrt{74}$, $\cos x = \frac{7}{74}\sqrt{74}$, $\cot x = -\frac{7}{5}$, $\sec x = \frac{1}{7}\sqrt{74}$, $\csc x = -\frac{1}{5}\sqrt{74}$.

16. $\sec z = 3$, z in first quadrant.

17. $\cot \phi = -\frac{1}{4}$, ϕ in second quadrant.

Ans. $\sin \phi = \frac{4}{17}\sqrt{17}$, $\cos \phi = -\frac{1}{17}\sqrt{17}$, $\tan \phi = -4$, $\sec \phi = -\sqrt{17}$, $\csc \phi = \frac{1}{4}\sqrt{17}$.

18. $\csc x = 8/5$, x in first quadrant.

In the following examples assume that x is an angle in the first quadrant.

19. Find algebraically the value of $(\sec^2 x - 1)\cot x$, given $\tan x = 3$. *Ans.* 3.

20. Find algebraically the value of $(\sin^2 x - \cos^2 x)\csc x$, given $\sin x = 1/2$.

21. Find the value of $(\tan^2 x + \cot^2 x)(\sec^2 x + \csc^2 x)$, given $\tan x = 2$. *Ans.* 425/16.

22. Find the value of $(1 - \csc^2 x)\tan x$, given $\sin x = 1/3$.

23. Find the value of $\cot^2 x - \sin^2 x + \tan^2 x - \cos^2 x$, given $\cos x = 1/3$. *Ans.* 57/8.

24. Find the value of $\dfrac{1 - \sin x \cos x}{1 + \sin x \cos x}$, given $\tan x = 2$.

25. Express each of the other functions of x in terms of $\sin x$.

Ans. $\sqrt{1-\sin^2 x}, \dfrac{\sin x}{\sqrt{1-\sin^2 x}}, \dfrac{\sqrt{1-\sin^2 x}}{\sin x}, \dfrac{1}{\sqrt{1-\sin^2 x}}, \dfrac{1}{\sin x}$.

26. Express each of the other functions of x in terms of (1) the cosine, (2) the tangent, (3) the secant of x.

Reduce the following expressions to equivalent expressions involving only $\sin x$.

27. $\sin x \tan x$.

28. $\dfrac{\cot x}{\csc x}$.

29. $\sec^2 x + \csc^2 x$.

30. $\tan^2 x + \cot^2 x$.

31. $\sec x\,(1 - \tan x)$.

32. $\dfrac{\cot x - 1}{\csc^2 x}$.

33. $\cos x \tan x$.

34. $\dfrac{\sec x - \tan x}{\csc x - \cot x}$.

35. $\dfrac{\cos x - \cot x}{\tan x - \csc x}$.

36. $\sin x \cot x$.

37. $\tan x + \cot x$.

38. $(\tan x - 1)(\cot x - 1)$.

39. $\sec x \csc x$.

40. $(\tan x - \sec x)(\tan x + \sec x)$.

Reduce the following expressions to expressions containing only $\tan x$.

41. $\csc^2 x - \sec^2 x$.

42. $\cos^2 x + 1$.

43. $\cot x \sec^2 x$.

44. $\cot x \sin x + \cos x \tan x$.

Find the value of each of the following expressions.

45. $\cos\theta \csc^2\theta$, when $\sin\theta = \frac{2}{3}$. *Ans.* $\pm \frac{3}{4}\sqrt{5}$.

46. $\dfrac{\sin x \cos x}{2\csc^2 x - 1}$, when $\tan x = 3$.

47. $\dfrac{\sin^2 y}{(\tan^2 y + 1)\cos y}$, when $\cos y = \frac{3}{5}$. *Ans.* $\frac{48}{125}$.

48. $\dfrac{\cot y}{\sec y \sqrt{\tan^2 y + 2}}$, when $\cos y = \frac{4}{5}$.

69. Identities and equations. — An *identity* is a statement of equality which is true for all values of the quantities involved. Thus the algebraic statement

$$x^2 - 2xy + y^2 = (x - y)^2$$

is an identity, because it is true for all values of x and y. The statement

$$\sin^2 x + \cos^2 x = 1$$

is a trigonometric identity, because it is true for any value of the angle x.

A *conditional equation* (or, for simplicity, an *equation*) is a statement of equality which is true only for certain particular values or sets of values of the quantities involved. Thus,

$$x^2 - 5x + 6 = 0$$

is a conditional equation, because it is untrue for most values of x, but is true only for $x = 2$ and $x = 3$. The trigonometric equation

$$\sin x = \cos x$$

is conditional, because it is true only for $x = \pi/4$, $x = 5\pi/4$, and for values of x coterminal with these.

The implication of an identity is merely that of a simple declarative sentence. It states that the quantity on the left-hand side is, for all values of the letters, equal to the quantity on the right-hand side, and the one quantity can always be reduced to or replace the other.

The implication of an equation, on the other hand, is that of a question. When we write $x^2 - 5x + 6 = 0$, we imply the question: For what values of x is this statement true? The solution of an equation, whether it be trigonometric or of some other form, consists in finding all of the values of the letters for which the equation is satisfied.

70. Trigonometric identities. — It is frequently necessary to be able to change a trigonometric expression from one form into another

form which is identically equal to it, but which is more convenient or suitable for the purpose in view. Sometimes these reductions are difficult to make. The most practicable method of learning how to make them is to practice in verifying or proving the truth of given trigonometric identities.

With this end in view, when a student is asked to prove an identity he should strive by one or more changes to transform one side of the identity into the form of the other side *which should be left unchanged.* If no other way can be found for proving the identity, the method may be suggested by expressing all of the functions in terms of sines and cosines, which may be done by means of the fundamental relations.

No general rule can be given for proving identities. Practice is the best teacher, and thorough familiarity with the fundamental relations the best tool. We must always keep in mind not only the expression which we are trying to modify but also the form to which we wish to reduce it so as to select the useful transformation and avoid mere futile fumbling. In general it is better to work upon the more complicated side of the identity and to try to simplify it to the form of the other side.

EXAMPLE 1. Prove the identity

$$\sin x = \frac{\tan x}{\sec x}.$$

The right-hand side is the more complicated, and we will therefore work with it, expressing it in terms of $\sin x$ and $\cos x$. We have

$$\frac{\tan x}{\sec x} = \frac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}} = \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} = \sin x,$$

which proves the identity.

EXAMPLE 2. Prove the identity

$$\tan^2 x - \sin^2 x = \sin^2 x \cdot \tan^2 x.$$

We shall modify the left-hand side, and write

$$\begin{aligned}\tan^2 x - \sin^2 x &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x,\\ &= \sin^2 x\left(\frac{1}{\cos^2 x} - 1\right),\\ &= \sin^2 x\,(\sec^2 x - 1),\\ &= \sin^2 x \cdot \tan^2 x,\end{aligned}$$

as we wished to show. Another method is to transform the right-hand side, as follows

$$\begin{aligned}\sin^2 x \cdot \tan^2 x &= (1 - \cos^2 x)\tan^2 x\\ &= \tan^2 x - \tan^2 x \cdot \cos^2 x\\ &= \tan^2 x - \frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x\\ &= \tan^2 x - \sin^2 x.\end{aligned}$$

EXERCISES

Prove the following identities.

1. $\tan x = \pm \dfrac{\sqrt{1 - \cos^2 x}}{\cos x}$.
2. $\dfrac{\sec^2 x}{\sin^2 x} = \csc^2 x + \sec^2 x$.
3. $(\sin x + \cos x)(\sec x - \csc x) = \tan x - \cot x$.
4. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x$.
5. $\csc^2 x\,(1 - \cos^2 x) = 1$.
6. $2 \sin x + \tan x = \dfrac{2 + \sec x}{\csc x}$.
7. $\dfrac{\tan x + \cot x}{\tan x - \cot x} = \dfrac{\tan^2 x + 1}{\tan^2 x - 1}$.
8. $\dfrac{\sec x - \cos x}{\sec x + \cos x} = \dfrac{\sec^2 x - 1}{\sec^2 x + 1}$.
9. $\sin x \tan x = \sec x - \cos x$.
10. $\sin^3 x + \cos^3 x = (\sin x + \cos x)(1 - \sin x \cos x)$.
11. $\tan x + \cot x = \sec x \csc x$.

12. $\sin^2 x \sec^2 x = \sec^2 x - 1.$
13. $\cos^2 x \csc^2 x = \csc^2 x - 1.$
14. $1 - \cos\theta = \dfrac{\sin^2\theta}{1 + \cos\theta}.$
15. $(1 + \cos x)(\csc x - \cot x) = \sin x.$
16. $\dfrac{1 + \tan\theta}{\sec\theta} = \sin\theta + \cos\theta.$
17. $\dfrac{1 - \cot\theta}{1 + \cot\theta} = \dfrac{\tan\theta - 1}{\tan\theta + 1}.$
18. $\dfrac{\sec\theta - 1}{\sec\theta + 1} = \dfrac{1 - \cos\theta}{1 + \cos\theta}.$
19. $\dfrac{1 + \tan^2\alpha}{1 + \cot^2\alpha} = \dfrac{\sin^2\alpha}{\cos^2\alpha}.$
20. $\sqrt{\dfrac{1 - \cos\alpha}{1 + \cos\alpha}} = \csc\alpha - \cot\alpha.$
21. $\csc^2 x + \cot^2 x = \csc^4 x - \cot^4 x.$
22. $\dfrac{\sin x}{\sec x} = \dfrac{1}{\tan x + \cot x}.$
23. $\dfrac{\sin y}{\tan y} = \dfrac{1}{\sec y}.$
24. $(1 - \cos y)(1 + \cos y) = \sin^2 y.$
25. $(\csc x - \cot x)(\csc x + \cot x) = 1.$
26. $\dfrac{\tan^2 x}{\sin^2 x} - \dfrac{\sin^2 x}{\cos^2 x} = 1.$
27. $\sin^2 y \cot^2 y + \cos^2 y \tan^2 y = 1.$
28. $\cos^2 y = \dfrac{\cot^2 y}{1 + \cot^2 y}.$
29. $\csc y - \sin y = \sin y \cot^2 y.$
30. $\dfrac{\sin^2\theta}{(1 - \cos\theta)^2} = \dfrac{(1 + \cos\theta)^2}{\sin^2\theta}.$
31. $\cot x + \cot y = \cot x \cot y (\tan x + \tan y).$
32. $(\sec x - \tan x)(\sec x + \tan x) = 1.$
33. $(\sin\alpha\cos\beta + \cos\alpha\sin\beta)^2 + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)^2 = 1.$
34. $(\sin x\cos y - \cos x\sin y)^2 + (\cos x\cos y + \sin x\sin y)^2 = 1.$

35. $(\cos^2 x - \sin^2 x)^2 + 4 \sin^2 x \cos^2 x = 1.$

36. $\dfrac{\tan x + \tan y}{1 - \tan x \tan y} \cdot \dfrac{\cot x \cot y - 1}{\cot x + \cot y} = 1.$

71. Trigonometric equations. — A solution of a trigonometric equation consists in finding all the angles which satisfy the equation. It is enough to give the values which satisfy the equation and lie between 0 and 2π (0° and 360°).

Usually we solve the equation for one or more trigonometric functions of the angle, and then find the corresponding values of the angle. A solution is partly a matter of algebra and partly a matter of trigonometry. The following suggestions apply to the *algebraic* part: Sometimes after arranging the equation so that we have a certain expression set equal to zero, we find that the expression can be factored. Then set each factor equal to zero and solve the simpler equations that result.

EXAMPLE 1. Solve the following equation for values of x from 0° to 360°.

$$\sin^2 x = \sin x.$$

Transposing and factoring, we get

$$\sin^2 x - \sin x = 0, \quad \text{or} \quad \sin x (\sin x - 1) = 0.$$

We must now set *each* of the factors equal to zero. For if either factor is equal to zero the equation holds true. We get

$$\sin x = 0 \quad \text{and} \quad \sin x - 1 = 0 \quad \text{or} \quad \sin x = 1.$$

The values of x from 0° to 360° for which $\sin x = 0$ are 0°, 180°, 360°. The only value of x in this range for which $\sin x = 1$ is 90°. The roots of the equation are therefore 0°, 90°, 180°, 360°.

In the course of the algebraic work if we have occasion to take the square root of both sides of the equation or to divide by a factor, we must take care not to throw away any roots.

EXAMPLE 2. In solving the equation

$$\sin^2 x = 1,$$

we must not only take $\sin x = 1$, giving $x = 90°$, but also $\sin x = -1$, giving $x = 270°$.

And in the equation

$$\sin^2 x = \sin x \cos x$$

if we divide by $\sin x$, getting

$$\sin x = \cos x,$$

we must also write $\sin x = 0$, since the given equation is satisfied if $\sin x = 0$.

On the other hand, if we square both sides of an equation (to get rid of radicals perhaps) or if we multiply it by a factor (in order to clear the equation of fractions), we must remember that we may have introduced factors that will lead to extraneous roots, that is, to values of the unknown that satisfy the derived equation but not the original given equation. The best test is to substitute the supposed roots in the given equation and reject any of them that fail to satisfy it.

On the side of trigonometry, if several functions of an angle are involved in the equation, it is suggested that all of the functions be expressed in terms of a single function by algebraic or geometric methods, and that while this is being done the use of radicals be avoided when possible. The methods are illustrated in the following examples.

EXAMPLE 3. Solve the equation

$$\cos^2 x + \sin x = 1$$

for values of x from 0° to 360°.

Transposing the first term, we get

$$\sin x = 1 - \cos^2 x,$$

or

$$\sin x = \sin^2 x,$$

which is the equation of Example 1.

EXAMPLE 4. Solve the equation

$$\cos x = -\sin x$$

for values of x from 0° to 360°.

Divide both sides by $\cos x$, getting

$$1 = -\frac{\sin x}{\cos x}, \quad \text{or} \quad \tan x = -1.$$

The values of x between 0° and 360° which satisfy this equation are $x = 135°$ and $x = 315°$. These results satisfy the given equation since

$$\cos 135° = -\tfrac{1}{2}\sqrt{2} = -\sin 135°,$$

and

$$\cos 315° = \tfrac{1}{2}\sqrt{2} = -\sin 315°.$$

Another method is to change the functions to sines, writing

$$\sqrt{1 - \sin^2 x} = -\sin x.$$

Squaring, we get

$$1 - \sin^2 x = \sin^2 x,$$

or

$$2\sin^2 x = 1 \quad \text{and} \quad \sin^2 x = \tfrac{1}{2},$$

or

$$\sin x = \pm\tfrac{1}{2}\sqrt{2}.$$

The angles for which this is satisfied are $x = 45°, 135°, 225°, 315°$. But on substituting in the original equation, we see that 45° and 225° do not satisfy, since $\sin 45° = \cos 45°$ and $\sin 225° = \cos 225°$. When we squared both sides of the equation we lost track of the signs, and introduced extraneous roots.

Example 5. Solve the equation

$$\sqrt{3}\tan x \sec x - 2 = \sec x - 2\sqrt{3}\tan x$$

for values of x from 0° to 360°.

We can write this in the form

$$\sqrt{3}\tan x(\sec x + 2) = \sec x + 2,$$

or

$$(\sqrt{3}\tan x - 1)(\sec x + 2) = 0.$$

Setting each factor equal to zero, we get

$$\sqrt{3}\tan x - 1 = 0, \quad \text{so that} \quad \tan x = 1/\sqrt{3}, \quad \text{and} \quad x = 30° \text{ or } 210°,$$

and

$$\sec x + 2 = 0, \quad \text{so that} \quad \sec x = -2, \quad \text{and} \quad x = 120° \text{ or } 240°.$$

The student should substitute these results, and see that they satisfy the given equation.

EXAMPLE 6. Solve the equation

$$\sec^2 x\,(1 + \cos x \tan x) = 2.$$

Expressing all of the functions in terms of sines and cosines, we get

$$\frac{1}{\cos^2 x}\left(1 + \cos x \frac{\sin x}{\cos x}\right) = 2$$

or

$$1 + \sin x = 2\cos^2 x.$$

Using only sines, we get

$$1 + \sin x = 2\,(1 - \sin^2 x)$$

or

$$2\sin^2 x + \sin x - 1 = 0.$$

We factor this, writing

$$(2\sin x - 1)(\sin x + 1) = 0.$$

This gives

$$2\sin x - 1 = 0, \quad \text{or} \quad \sin x = \tfrac{1}{2}, \quad \text{so that} \quad x = 30° \text{ or } 150°,$$

and

$$\sin x + 1 = 0, \quad \text{or} \quad \sin x = -1, \quad \text{so that} \quad x = 270°.$$

The results $x = 30°$ and $x = 150°$ satisfy the equation, but we must reject 270°, for the original equation involves $\tan x$ and $\sec x$, and $\tan 270°$ and $\sec 270°$ do not exist. We introduced the extraneous result when we multiplied both sides of the equation by $\cos^2 x$, which is zero when $x = 270°$.

EXERCISES

Solve the following equations for positive (or zero) angles less than 360°.

1. $\sin^2\theta = \frac{1}{4}$. *Ans.* 30°, 150°, 210°, 330°.
2. $\cos^2\theta = \frac{1}{4}$.
3. $\tan^2\theta = 1$. *Ans.* 45°, 135°, 225°, 315°.
4. $\sec^2\theta = 1$.
5. $(\sin\theta + 1)(2\cos\theta - 1) = 0$. *Ans.* 60°, 270°, 300°.
6. $(2\cos\theta + \sqrt{3})(\sqrt{2}\sin\theta - 1) = 0$.
7. $(\tan^2 x - 3)(\csc x - 2) = 0$.

Ans. 30°, 60°, 120°, 150°, 240°, 300°.

8. $(\cot x + 1)(3 \sec^2 x - 4) = 0$.
9. $\sqrt{2} \tan \theta \sin \theta - \tan \theta = 0$. *Ans.* 0°, 45°, 135°, 180°, 360°.
10. $2 \sec \theta \sin \theta - \sec \theta = 0$.
11. $2 \tan^2 \theta - \sec^2 \theta = 0$. *Ans.* 45°, 135°, 225°, 315°.
12. $(4 \cos^2 \theta - 3) \csc^2 \theta = 0$.
13. $1 - \sin x = \sqrt{3} \cos x$. *Ans.* 90°, 330°.
14. $1 + \cos x + \sqrt{3} \sin x = 0$.
15. $\csc x - 1 = \cot x$. *Ans.* 90°.
16. $\cos x - \sin x = 1$.
17. $2 \sin^2 x + \sin x - 1 = 0$. *Ans.* 30°, 150°, 270°.
18. $2 \sin^2 x = \cos x + 1$.
19. $\csc^2 x - \cot x = 1$. *Ans.* 45°, 90°, 225°, 270°.
20. $\sec^2 x = 1 - \sqrt{3} \tan x$.
21. $\tan^2 x + 3 \csc^2 x = 7$.
Ans. 45°, 60°, 120°, 135°, 225°, 240°, 300°, 315°.
22. $3 \cot^2 x + \sec^2 x = 5$.
23. $2 \cot x + \cos x \cot x = 2 \cot x \sin^2 x$.
Ans. 90°, 120°, 240°, 270°.
24. $2 \sin \theta \tan \theta = 2 \cos \theta$.
25. $(2 \sin^2 x - 1)(\sin x - 3) = 0$. *Ans.* 45°, 135°, 225°, 315°.
26. $(\sqrt{2} \cos x - 2)(\cos^2 x - 4) = 0$.
27. $\sqrt{3} \tan x - 1 = \sec x$. *Ans.* 60°, 180°.
28. $5 (\csc x - 1) = 2 \cos x \cot x$.
29. $\tan x + \sqrt{3} \cot x = \sqrt{3} + 1$. *Ans.* 45°, 225°, 60°, 240°.
30. $4 \csc^2 x - 7 \cot^2 x = 3$.

Prove the following identities.

31. $\tan x \sec x = \sin^3 x \sec^2 x + \sin x$.
32. $\cot x \csc x = \dfrac{1}{\sec x - \cos x}$.
33. $\tan^2 x - \sin^2 x = \sin^4 x \sec^2 x$.
34. $\dfrac{1 + \cos x}{1 - \cos x} = 1 + \dfrac{2 \cos x (1 + \cos x)}{\sin^2 x}$.
35. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x} = 2 \sec x$.

36. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x.$

37. $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x \cos^2 x.$

38. $(1 - \sin x)(\sec x + \tan x) = \cos x.$

39. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta.$

40. $(1 - \tan\theta)(1 - \cot\theta) = 2 - \sec\theta \csc\theta.$

41. $\cos\alpha \csc^2\alpha \tan\alpha - \sin\alpha = \cos^2\alpha \csc\alpha.$

42. $\dfrac{\tan\alpha - \sin\alpha}{\sin^3\alpha} = \dfrac{\sec\alpha}{1 + \cos\alpha}.$

43. $\dfrac{\cot\alpha - \tan\alpha}{\cos\alpha - \sin\alpha} = \dfrac{\cos\alpha + \sin\alpha}{\cos\alpha \sin\alpha}.$

44. $\sqrt{\dfrac{1 - \sin\alpha}{1 + \sin\alpha}} = \sec\alpha - \tan\alpha.$

45. $\dfrac{\sin\theta}{\cot\theta} + \dfrac{\cos\theta}{\tan\theta} = (1 - \sin\theta\cos\theta)(\sec\theta + \csc\theta).$

46. $\dfrac{\tan x - \tan y}{\sec x + \sec y} = \dfrac{\sec x - \sec y}{\tan x + \tan y}.$

47. $\dfrac{\cot x + \cot y}{\csc x - \csc y} = \dfrac{\csc x + \csc y}{\cot x - \cot y}.$

In the following examples reduce the left-hand side to the form of the right-hand side.

48. $\tan x + x\sec^2 x - \sin x \sec x - x = x\tan^2 x.$

49. $\dfrac{a\sin x\cos x - b\sin x\cos x}{a\sin^2 x + b\cos^2 x} = \dfrac{(a - b)\tan x}{a\tan^2 x + b}.$

50. $\sec^m x - \sec^{m-2} x = \sec^{m-2} x \tan^2 x.$

51. $6\tan^2 x \sec^3 x + 2\tan^4 x \sec x + \sec^3 x + \tan^2 x \sec x - \sec x$
$= 8\tan^2 x \sec^3 x.$

52. $\sec^2 x(\tan x - 3\cot x) + \tan x(\sec^2 x + 3\csc^2 x)$
$= 2\sec^2 x \tan x.$

53. $\dfrac{a(a\cos x - b\sin x)}{a\sin x + b\cos x} + b = \dfrac{a^2 + b^2}{a\tan x + b}.$

54. $\sec^6 x = (\tan^2 x + 1)^2 \sec^2 x.$

55. $\csc^4 x = (\cot^2 x + 1)\csc^2 x.$

CHAPTER VIII

FUNCTIONS OF TWO ANGLES

72. Sin $(\alpha + \beta)$ and cos $(\alpha + \beta)$. — The sine of an angle which is the sum of two angles is not, in general, equal to the sum of the sines of the two angles.

A single example is enough to show this. Let $\alpha = 30°$ and $\beta = 60°$. Then

$$\sin \alpha + \sin \beta = \sin 30° + \sin 60° = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3} = \tfrac{1}{2}(1 + \sqrt{3}).$$

But this is not equal to

$$\sin (\alpha + \beta) = \sin (30° + 60°) = \sin 90° = 1.$$

We proceed now to obtain formulas for the sine and the cosine of the sum of two angles in terms of the functions of the separate angles; that is, for $\sin (\alpha + \beta)$ and $\cos (\alpha + \beta)$ in terms of the functions of α and β. For the present we shall prove the formulas only for the case in which α and β are positive, acute angles, and in Article **73** will prove that the same formulas are true for all values of α and β whatsoever.

Suppose, then, that α and β are any two positive, acute angles. Then $(\alpha + \beta)$ is less than 180°. Place the angle α in standard position with reference to a system of coördinate axes. Place the angle β with its vertex at the origin and its initial side, OQ, on the terminal side of α. The angle $(\alpha + \beta)$ is then also in standard position. Let P be any convenient point on the terminal side of $(\alpha + \beta)$. From P draw the line PQ perpendicular to the common side of α and β. Draw PA and QB perpendicular to the x-axis, and draw QC perpendicular to AP. Then the angle CPQ is equal to α, since its sides are respectively perpendicular to the sides of α.

From the right triangle BOQ, we see that

(1) $BQ = OQ \cdot \sin \alpha$, and (2) $OB = OQ \cdot \cos \alpha$.

From the right triangle PCQ, we find that

(3) $CQ = QP \cdot \sin \alpha$ and (4) $CP = QP \cdot \cos \alpha$.

And from the right triangle OQP, we obtain

(5) $\frac{QP}{OP} = \sin \beta$ and (6) $\frac{OQ}{OP} = \cos \beta$.

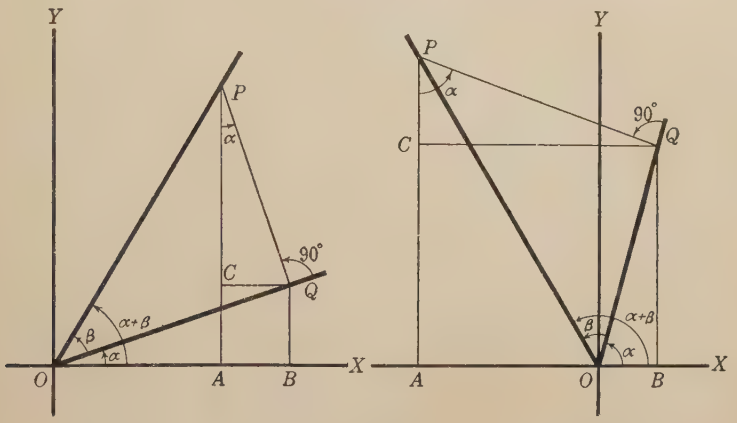

FIG. 54

From the definition of the sine of an angle we have

$$\sin (\alpha + \beta) = \frac{AP}{OP} = \frac{AC + CP}{OP} = \frac{AC}{OP} + \frac{CP}{OP} = \frac{BQ}{OP} + \frac{CP}{OP}.$$

From (1) and (4) this yields

$$\sin (\alpha + \beta) = \frac{OQ}{OP} \sin \alpha + \frac{QP}{OP} \cos \alpha.$$

Then, applying (5) and (6), we get the desired formula:

I. $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$

Likewise, by the definition of the cosine of an angle,

$$\cos (\alpha + \beta) = \frac{OA}{OP} = \frac{OB - AB}{OP} = \frac{OB}{OP} - \frac{AB}{OP} = \frac{OB}{OP} - \frac{CQ}{OP}.$$

By the use of (2) and (3) and then of (5) and (6) we get

$$\cos(\alpha + \beta) = \frac{OQ}{OP}\cos\alpha - \frac{QP}{OP}\sin\alpha,$$

and, finally,

II. $$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

These two laws may be stated as follows:

The sine of the sum of two angles is equal to the sine of one of the angles multiplied by the cosine of the second, plus the cosine of the first angle times the sine of the second.

The cosine of the sum of two angles is equal to the product of the cosines of the two angles minus the product of their sines.

It is important that the student learn these laws and all of the others in this chapter and be able to express them either in formulas or in words.

EXAMPLE. Given $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}$, find $\sin 75°$ and $\cos 75°$.

$$\sin 75° = \sin(30° + 45°) = \sin 30°\cos 45° + \cos 30°\sin 45°.$$
$$= \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3}\cdot\tfrac{1}{2}\sqrt{2} = \tfrac{1}{4}\sqrt{2}\,(1 + \sqrt{3})$$
$$= \frac{1.414 \times 2.732}{4} = .966.$$

$$\cos 75° = \cos(30° + 45°) = \cos 30°\cos 45° - \sin 30°\sin 45°$$
$$= \tfrac{1}{2}\sqrt{3}\cdot\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{2} = \tfrac{1}{4}\sqrt{2}\,(\sqrt{3} - 1)$$
$$= \frac{1.414 \times .732}{4} = .259.$$

★**73. Extension of addition formulas to all angles.** — We have proved formulas I and II of Article **72** if α and β are any two *acute* angles. We wish to prove that they are true for all values of α and β.

I. We shall first show* that *if α and β are any two angles* (acute or not) *for which these formulas hold, the formulas are still true if either α or β is either increased or decreased by* 90°. In particular, suppose

* The reader who is familiar with the process of proof by mathematical induction will recognize this as an example of that process.

that α is increased by 90°. Assume, then, that α and β are any two angles such that

(1) $$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

(2) $$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

and, on this hypothesis, let us try to prove that

(3) $$\sin(\alpha' + \beta) = \sin\alpha'\cos\beta + \cos\alpha'\sin\beta,$$

(4) $$\cos(\alpha' + \beta) = \cos\alpha'\cos\beta - \sin\alpha'\sin\beta,$$

where $$\alpha' = \alpha + 90°.$$

We have

(5) $$\sin(\alpha' + \beta) = \sin(\alpha + \beta + 90°) = \cos(\alpha + \beta) \quad \text{(Article 53.)}$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \text{by equation (2).}$$

But $$\cos\alpha = \cos(\alpha' - 90°) = \sin\alpha',$$

and $$\sin\alpha = \sin(\alpha' - 90°) = -\cos\alpha'. \quad \text{(Article 54.)}$$

Consequently, from (5),

(6) $$\sin(\alpha' + \beta) = \sin\alpha'\cos\beta + \cos\alpha'\sin\beta,$$

which is equation (3). In a similar way,

$$\cos(\alpha' + \beta) = \cos(\alpha + \beta + 90°) = -\sin(\alpha + \beta)$$
$$= -[\sin\alpha\cos\beta + \cos\alpha\sin\beta] \quad \text{by equation (1),}$$
$$= -[-\cos\alpha'\cos\beta + \sin\alpha'\sin\beta],$$

which gives equation (4).

It is left to the student to prove that if (1) and (2) are true, then (3) and (4) must be true, where

$$\alpha' = \alpha - 90°,$$

and also to show, by Article **53**, that equations (1) and (2) are true when either α or β instead of being actually acute is equal to 90°.

II. Now suppose that α' and β' are any two angles whatsoever. We can then find two angles, α and β, either acute or right angles, such that α' can be obtained from α by adding or subtracting $90°$ a certain number of times and β' can be obtained from β by adding or subtracting $90°$ a certain number of times. Therefore, if the formulas (1) and (2) hold for this α and β, the formulas must also hold for α' and β'. For, holding β fast, we can replace α by α' by making successive additions or subtractions of $90°$, and after each such change the formulas will still be true. Then β, in a similar way, can be replaced by β'.

But the formulas (1) and (2) are true for α and β since α and β are acute or right angles. Therefore the formulas are true for α' and β', that is, for any angles.

74. Tan $(\alpha + \beta)$ and cot $(\alpha + \beta)$. — By means of I and II, we can write

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing numerator and denominator of this fraction by $\cos\alpha\cos\beta$, we get

$$\tan(\alpha+\beta) = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}},$$

or

III. $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta};$$

the tangent of the sum of two angles is equal to the sum of the tangents of the angles divided by 1 *minus the product of their tangents.*

It is left to the student to prove the formula

IV. $$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$$

EXAMPLE.

$$\tan 75^\circ = \tan (45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$= \frac{4 + 2\sqrt{3}}{3 - 1} = 2 + \sqrt{3} = 3.73.$$

75. Functions of $(\alpha - \beta)$. — Since the formulas for functions of $(\alpha + \beta)$ are true for all values of α and β, we may replace β by $-\beta$. Since $\cos(-\beta) = \cos\beta$, while $\sin(-\beta) = -\sin\beta$, $\tan(-\beta) = -\tan\beta$, and $\cot(-\beta) = -\cot\beta$, the terms in formulas I, II, III, and IV which contain $\sin\alpha$, $\tan\alpha$, or $\cot\alpha$ will be reversed in sign, while the other terms will remain unchanged. We get

V. $$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

VI. $$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

VII. $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

VIII. $$\cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}.$$

EXAMPLE.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}\sqrt{2}(\sqrt{3} - 1) = .259.$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ)$$
$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}\sqrt{2}(\sqrt{3} + 1) = .966.$$

$$\tan 15^\circ = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{3 - 1} = 2 - \sqrt{3} = .268.$$

EXERCISES

Suppose that α and β are two positive acute angles. Find the sine, cosine, and tangent of $(\alpha + \beta)$ and $(\alpha - \beta)$, under each of the following suppositions.

1. $\sin \alpha = 2/3, \cos \beta = 3/5.$

Solution: We first find the other functions of α and β. By either algebraic or geometric methods we find $\cos \alpha = \sqrt{5}/3$; $\tan \alpha = 2/\sqrt{5} = \frac{2}{5}\sqrt{5}$.

$$\sin \beta = 4/5; \qquad \tan \beta = 4/3.$$

Then

$$\sin (\alpha + \beta) = \frac{2}{3} \cdot \frac{3}{5} + \frac{\sqrt{5}}{3} \cdot \frac{4}{5} = \frac{2}{15}(3 + 2\sqrt{5}).$$

$$\sin (\alpha - \beta) = \frac{2}{3} \cdot \frac{3}{5} - \frac{\sqrt{5}}{3} \cdot \frac{4}{5} = \frac{2}{15}(3 - 2\sqrt{5}).$$

$$\cos (\alpha + \beta) = \frac{\sqrt{5}}{3} \cdot \frac{3}{5} - \frac{2}{3} \cdot \frac{4}{5} = \frac{1}{15}(3\sqrt{5} - 8).$$

$$\cos (\alpha - \beta) = \frac{\sqrt{5}}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{4}{5} = \frac{1}{15}(3\sqrt{5} + 8).$$

$$\tan (\alpha + \beta) = \frac{\frac{2}{\sqrt{5}} + \frac{4}{3}}{1 - \frac{2}{\sqrt{5}} \cdot \frac{4}{3}} = \frac{6 + 4\sqrt{5}}{3\sqrt{5} - 8} = -\tfrac{1}{19}(108 + 50\sqrt{5}),$$

a result obtainable from the values of $\sin (\alpha + \beta)$ and $\cos (\alpha + \beta)$ already found.

$$\tan (\alpha - \beta) = \frac{6 - 4\sqrt{5}}{3\sqrt{5} + 8} = \tfrac{1}{19}(108 - 50\sqrt{5}).$$

2. $\sin \alpha = 4/5, \cos \beta = 5/13.$

3. $\tan \alpha = 2, \cot \beta = 3/2.$

Ans. $\sin (\alpha + \beta) = \frac{8}{65}\sqrt{65}$, $\sin (\alpha - \beta) = \frac{4}{65}\sqrt{65}$,
$\cos (\alpha + \beta) = -\frac{1}{65}\sqrt{65}$, $\cos (\alpha - \beta) = \frac{7}{65}\sqrt{65}$,
$\tan (\alpha + \beta) = -8$, $\tan (\alpha - \beta) = \frac{4}{7}$.

4. $\tan \alpha = 3, \sec \beta = 2.$

5. Find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, given that $\cos(\alpha+\beta) = 3/4$ and $\sin\beta = 1/3$, and that all the angles are acute.

Ans. $\sin\alpha = \frac{1}{12}(\sqrt{56} - 3)$, $\cos\alpha = \frac{1}{12}(3\sqrt{8} + \sqrt{7})$,
$\tan\alpha = \frac{1}{65}(27\sqrt{7} - 16\sqrt{8})$.

6. Find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, given that $\cos(\alpha-\beta) = 3/4$ and $\sin\beta = 1/4$, and that all the angles are acute.

Assuming that α and β are acute, in what quadrant does $(\alpha+\beta)$ lie in each of the following cases?

7. $\sin\alpha = 3/4$, $\sin\beta = 2/3$. *Ans.* Second.

8. $\sin\alpha = 2/3$, $\sin\beta = 2/3$.

9. $\cos\alpha = 2/3$, $\cos\beta = 1/2$. *Ans.* Second.

10. $\cos\alpha = 1/2$, $\sin\beta = 3/5$.

11. Find $\sin 120°$ from functions of $30°$ and $90°$.

12. Find $\cos 150°$ from functions of $180°$ and $30°$.

13. Show that

$$\begin{aligned}\sin(\alpha+\beta+\gamma) &= \sin[(\alpha+\beta)+\gamma]\\ &= \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma\\ &\quad + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma.\end{aligned}$$

14. Show that

$$\begin{aligned}\cos(\alpha+\beta+\gamma) &= \cos\alpha\cos\beta\cos\gamma - \sin\alpha\cos\beta\sin\gamma\\ &\quad - \cos\alpha\sin\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma.\end{aligned}$$

15. Show that

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}.$$

Find the values of the following expressions.

16. $\sin(30° + \alpha)$.

17. $\cos(\alpha + 45°)$. *Ans.* $\frac{1}{2}\sqrt{2}(\cos\alpha - \sin\alpha)$.

18. $\tan(\alpha + 60°)$.

Show that

19. $\cos(x + 30°) - \cos(x - 30°) = -\sin x.$
20. $\cos(x + 45°) + \cos(x - 45°) = \sqrt{2}\cos x.$
21. $\sin(x + 60°) + \sin(x - 60°) = \sin x.$
22. $\sin(x + 30°) + \cos(x + 60°) = \cos x.$
23. $\cos(\alpha + \beta)\cos(\alpha - \beta) + \sin(\alpha + \beta)\sin(\alpha - \beta) = 1 - 2\sin^2\beta.$
24. $\cos(\alpha + \beta)\cos(\alpha - \beta) - \sin(\alpha + \beta)\sin(\alpha - \beta) = 2\cos^2\alpha - 1.$
25. $\tan(\alpha + 45°) + \cot(\alpha - 45°) = 0.$
26. $\dfrac{\tan(45° + \alpha)}{\tan(45° - \alpha)} = \dfrac{(1 + \tan\alpha)^2}{(1 - \tan\alpha)^2}.$
27. $\cos(x + y)\cos y - \cos(x + z)\cos z$
$= \sin(x + z)\sin z - \sin(x + y)\sin y.$
28. $\sin(x + y)\cos y + \cos(x + z)\sin z$
$= \sin(x + z)\cos z + \cos(x + y)\sin y.$

76. Functions of twice an angle. — If we replace β by α in Formulas I, II, III, and IV, $(\alpha + \beta)$ becomes equal to 2α, and we have

IX. $\sin 2\alpha = 2\sin\alpha\cos\alpha.$

X. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$= 1 - 2\sin^2\alpha$
$= 2\cos^2\alpha - 1.$

XI. $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}.$

XII. $\cot 2\alpha = \dfrac{\cot^2\alpha - 1}{2\cot\alpha}.$

We have three forms for $\cos 2\alpha$ obtained by setting

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = (1 - \sin^2\alpha) - \sin^2\alpha = 1 - 2\sin^2\alpha,$$

and

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1 - \cos^2\alpha) = 2\cos^2\alpha - 1.$$

The formulas should be expressed in words, as, for example, *the cosine of twice an angle is equal to the square of the cosine of the angle*

minus the square of the sine of the angle; or to 1 *minus twice the square of the sine of the angle; or to twice the square of the cosine of the angle minus* 1.

EXAMPLE. Verify the formulas for the case $\alpha = 30°$.

By IX, $\sin 60° = \sin (2 \times 30°) = 2 \sin 30° \times \cos 30°$

$= 2 \times \frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{1}{2}\sqrt{3}.$

$\cos 60° = \cos (2 \times 30°) = \cos^2 30° - \sin^2 30° = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$

$$\tan 60° = \tan (2 \times 30°) = \frac{2 \tan 30°}{1 - \tan^2 30°} = \frac{\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3}.$$

$$\cot 60° = \cot (2 \times 30°) = \frac{\cot^2 30° - 1}{2 \cot 30°} = \frac{3 - 1}{2\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

77. Functions of half an angle. — From the results of the preceding article we obtain formulas for the functions of half an angle in terms of the functions of the whole angle. Since, by Article **76**, the cosine of twice an angle is equal to 1 minus twice the square of the sine of the angle, by taking $\alpha/2$ as "the angle" and α as "twice the angle," we have

$$\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}.$$

Therefore

$$2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha,$$

and

XIII. $$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$$

Likewise from the third form for the cosine of twice an angle,

$$\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1,$$

and

XIV. $$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}.$$

On dividing XIII by XIV, we find

XV. $$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

XVI. $$\cot\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}.$$

The student should state these relations in words.

EXAMPLE 1. Verify these formulas for the angle $\alpha = 60°$. Now $\alpha/2 = 30°$, and

$$\sin 30° = \sin\frac{60°}{2} = \sqrt{\frac{1-\cos 60°}{2}} = \sqrt{\frac{1-\frac{1}{2}}{2}} = \sqrt{\tfrac{1}{4}} = \tfrac{1}{2},$$

$$\cos 30° = \sqrt{\frac{1+\cos 60°}{2}} = \sqrt{\frac{1+\frac{1}{2}}{2}} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3},$$

$$\tan 30° = \sqrt{\frac{1-\cos 60°}{1+\cos 60°}} = \sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}} = \sqrt{\tfrac{1}{3}} = \tfrac{1}{3}\sqrt{3}.$$

EXAMPLE 2. Prove the identity

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

We can write

$$\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin 2x\cos x + \cos 2x\sin x \quad \text{(By I)} \\
&= (2\sin x\cos x)\cos x + (1 - 2\sin^2 x)\sin x \quad \text{(By IX and X)} \\
&= 2\sin x\cos^2 x + \sin x - 2\sin^3 x \\
&= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\
&= 3\sin x - 4\sin^3 x,
\end{aligned}$$

as we wished to show.

EXAMPLE 3. Prove the identity

$$\tan^2\alpha = 4\sin^2\alpha\csc^2 2\alpha - 1.$$

The right-hand side is the more complicated, so we simplify it. The chief difficulty is that the identity contains functions of two different angles, α and 2α. When functions of two angles are present in an equation, we usually try to reduce them to functions of one angle. In this example we express $\csc^2 2\alpha$ in terms of functions of α. We write

$$\begin{aligned}
4\sin^2\alpha\csc^2 2\alpha - 1 &= \frac{4\sin^2\alpha}{\sin^2 2\alpha} - 1 \\
&= \frac{4\sin^2\alpha}{(2\sin\alpha\cos\alpha)^2} - 1 \\
&= \frac{1}{\cos^2\alpha} - 1 = \sec^2\alpha - 1 = \tan^2\alpha.
\end{aligned}$$

EXERCISES

Find the values of the following functions without using the tables.

1. $\sin 90°$, from functions of $45°$.
2. $\tan 120°$, from functions of $60°$.
3. $\sin 22\frac{1}{2}°$. *Ans.* $\frac{1}{2}\sqrt{2-\sqrt{2}} = .38.$
4. $\cos 22\frac{1}{2}°$.
5. $\tan 15°$. *Ans.* $2-\sqrt{3} = .268.$
6. $\sin 15°$.
7. $\sin 67\frac{1}{2}°$. *Ans.* $\frac{1}{2}\sqrt{2+\sqrt{2}} = .924.$
8. $\cos 67\frac{1}{2}°$.

Prove the following identities.

9. $\cos 3x = 4\cos^3 x - 3\cos x.$
10. $4\sin^2 x \cos^2 x = 1 - \cos^2 2x.$
11. $\cos^2 x = \sin^2 x + \cos 2x.$
12. $\cos 4x = 1 - 8\sin^2 x \cos^2 x.$
13. $\sec 2x = \dfrac{1}{2\cos^2 x - 1}.$
14. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}.$
15. $\sec 2x = \dfrac{\csc^2 x}{\csc^2 x - 2}.$
16. $2\csc 2x = \sec x \csc x.$
17. $2\cot 2x = 2\cos x \csc x - \sec x \csc x.$
18. $\cos^4 x - \sin^4 x = \cos 2x.$
19. $1 - \sin 2x = (\sin x - \cos x)^2.$
20. $\dfrac{1 - \sin 2x}{\cos 2x} = \dfrac{1 - \tan x}{1 + \tan x}.$
21. $\dfrac{1 + \tan^2 x}{1 - \tan^2 x} = \sec 2x.$
22. $\cos 5x \cos x - \sin 5x \sin x = \cos 6x.$
23. $\cos 2x \cos x + \sin 2x \sin x = \cos x.$
24. $\dfrac{\cos 3x}{\sin x} + \dfrac{\sin 3x}{\cos x} = 2\cot 2x.$

25. $\dfrac{\cos 3x}{\sin x} - \dfrac{\sin 3x}{\cos x} = 4 \cot 4x \cos 2x.$

26. $\dfrac{1 + \tan x}{1 - \tan x} = \sec 2x + \tan 2x.$

27. $\dfrac{2 \cot x}{1 + \cot^2 x} = \sin 2x.$

28. $\dfrac{\sec x - 1}{\sec x + 1} = \sec^2 \dfrac{x}{2} - 1.$

29. $\dfrac{\sec x + 1}{\sec x - 1} = \csc^2 \dfrac{x}{2} - 1.$

30. $\tan 2x = \tan x \,(1 + \sec 2x).$

31. $\csc x = \tfrac{1}{2}\left(\tan \dfrac{x}{2} + \cot \dfrac{x}{2}\right).$

32. $\sec^2 \dfrac{x}{2} = \dfrac{2 \tan x}{\tan x + \sin x}.$

33. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2.$

34. $2 \cot 2x = \cot x - \tan x.$

35. $\dfrac{1 - \cos 2x}{\sin 2x} = \tan x.$

36. $\dfrac{\sec 2x - 1}{\sec 2x + 1} = \tan^2 x.$

37. $\csc x - \cot x = \tan \dfrac{x}{2}.$

38. $\dfrac{\sin x}{1 + \cos x} = \tan \dfrac{x}{2}.$

39. $\sin 2x = 2 \sin^3 x \cos x + 2 \sin x \cos^3 x.$

40. $4 (\sin^6 x + \cos^6 x) = 4 - 3 \sin^2 2x.$

78. Product formulas. — From the formulas

I. $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta,$

II. $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$

V. $\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta,$

VI. $\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta,$

we get

$$(1)\quad \cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta,$$
$$(2)\quad \cos(\alpha+\beta)+\cos(\alpha-\beta)=\ \ 2\cos\alpha\cos\beta,$$
$$(3)\quad \sin(\alpha+\beta)+\sin(\alpha-\beta)=\ \ 2\sin\alpha\cos\beta,$$
$$(4)\quad \sin(\alpha+\beta)-\sin(\alpha-\beta)=\ \ 2\cos\alpha\sin\beta.$$

From equations (1), (2), and (3) we obtain the *product formulas:*

$$\textbf{XVII.}\quad \sin\alpha\sin\beta=\tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)],$$
$$\textbf{XVIII.}\quad \cos\alpha\cos\beta=\tfrac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)],$$
$$\textbf{XIX.}\quad \sin\alpha\cos\beta=\tfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)].$$

For the purposes of this book the formulas of this Article are less important than those of Article **79**, and they may be omitted. In other fields of mathematics they are important.

These formulas should be stated in words. Thus, for XVII, we can say: *The product of the sines of two angles is equal to one-half the difference of the cosine of the difference of the angles and the cosine of the sum of the angles.*

Example. Verify these formulas in the case $\alpha = 30°$, $\beta = 60°$.

By XIII,

$$\sin 30°\sin 60° = \tfrac{1}{2}[\cos(-30°)-\cos 90°] \quad\text{or}\quad \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{3} = \tfrac{1}{2}[\tfrac{1}{2}\sqrt{3}-0];$$

by XIV,

$$\cos 30°\cos 60° = \tfrac{1}{2}[\cos 90°+\cos(-30°)] \quad\text{or}\quad \tfrac{1}{2}\sqrt{3}\cdot\tfrac{1}{2} = \tfrac{1}{2}[0+\tfrac{1}{2}\sqrt{3}];$$

by XV,

$$\sin 30°\cos 60° = \tfrac{1}{2}[\sin 90°+\sin(-30°)] \quad\text{or}\quad \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{2}[1-\tfrac{1}{2}].$$

79. Sums of functions. — In order to obtain formulas for the sum or difference of the sines or cosines of two angles, in equations (1) to (4) of Article **78**, we will set

$$\alpha+\beta = x$$
$$\alpha-\beta = y.$$

Adding these two equations, we get $2\alpha = x + y$, and on subtracting, $2\beta = x - y$. Therefore

$$\alpha = \tfrac{1}{2}(x + y), \quad \beta = \tfrac{1}{2}(x - y).$$

Using these values in equations (3), (4), (2), (1), we get:

XX. $\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y).$

XXI. $\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y).$

XXII. $\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y).$

XXIII. $\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y).$

The formulas should be stated in words. Thus, for XX: *The sum of the sines of two angles is equal to twice the sine of half the sum of the angles times the cosine of half the difference of the angles.*

EXAMPLE 1. Prove the identity

$$2 \sin 3z \cos 7z + \sin 4z = \sin 10z.$$

From XIX, using $\alpha = 3z$, $\beta = 7z$, we have

$$\sin 3z \cos 7z = \tfrac{1}{2}[\sin 10z + \sin(-4z)].$$

Therefore

$$2 \sin 3z \cos 7z + \sin 4z = \sin 10z - \sin 4z + \sin 4z = \sin 10z.$$

EXAMPLE 2. Prove the identity

$$\frac{\sin 7z + \sin 3z}{\cos 7z + \cos 3z} = \tan 5z.$$

Applying XX and XXII to numerator and denominator of the left-hand side, we find

$$\frac{\sin 7z + \sin 3z}{\cos 7z + \cos 3z} = \frac{2 \sin \tfrac{1}{2}(7z + 3z) \cos \tfrac{1}{2}(7z - 3z)}{2 \cos \tfrac{1}{2}(7z + 3z) \cos \tfrac{1}{2}(7z - 3z)}$$

$$= \frac{2 \sin 5z \cos 2z}{2 \cos 5z \cos 2z} = \frac{\sin 5z}{\cos 5z} = \tan 5z.$$

EXAMPLE 3. Solve the equation

$$\sin 2x + \sqrt{2} \cos x = 0$$

for all values of x from 0° to 360°.

We reduce this so that it will contain only the single angle x:

$$2 \sin x \cos x + \sqrt{2} \cos x = 0$$

or

$$\cos x (2 \sin x + \sqrt{2}) = 0.$$

Hence

$$\cos x = 0, \quad \text{so that} \quad x = 90° \text{ or } 270°, \text{ and}$$
$$2 \sin x + \sqrt{2} = 0 \quad \text{or} \quad \sin x = -\tfrac{1}{2}\sqrt{2}, \quad \text{so that} \quad x = 225° \text{ or } 315°.$$

These four values of x satisfy the original equation.

EXAMPLE 4. Solve the equation

$$\sin 3x + \sin 2x + \sin x = 0$$

for all values of x from 0° to 360°.

We write the equation

$$(\sin 3x + \sin x) + \sin 2x = 0,$$

and apply formula XX, getting

$$2 \sin 2x \cos x + \sin 2x = 0$$

or

$$\sin 2x (2 \cos x + 1) = 0.$$

Hence

$$2 \cos x + 1 = 0 \quad \text{or} \quad \cos x = -1/2,$$

which gives $x = 120°$ and $x = 240°$.

Also

$$\sin 2x = 0.$$

In order to get all of the values of x which satisfy this equation and which lie between 0° and 360°, we must find all of the values of $2x$ which satisfy the equation and lie between 0° and 720°. They are

$$2x = 0°, 180°, 360°, 540°, 720°,$$

so that

$$x = 0°, 90°, 180°, 270°, 360°.$$

The complete solution therefore is

$$x = 0°, 90°, 120°, 180°, 240°, 270°, 360°.$$

EXERCISES

Express the following products as sums or differences of two functions.

1. $2 \sin 7\alpha \sin 3\alpha$. *Ans.* $\cos 4\alpha - \cos 10\alpha$.
2. $2 \sin 2\alpha \sin \alpha$.
3. $2 \cos \frac{5\alpha}{2} \cos \frac{\alpha}{2}$. *Ans.* $\cos 3\alpha + \cos 2\alpha$.
4. $2 \cos (x + y) \cos (x - y)$.
5. $2 \sin (180° + \alpha) \cos (180° - \alpha)$.
   Ans. $\sin 2\alpha + \sin 360° = \sin 2\alpha$.
6. $\sin \frac{2x}{3} \cos \frac{x}{3}$.

Express each of the following quantities as a single term.

7. $\sin x + \sin (x + y)$. *Ans.* $2 \sin \left(x + \frac{y}{2}\right) \cos \frac{y}{2}$.
8. $\sin 5x + \sin x$.
9. $\cos 5x + \cos 3x$. *Ans.* $2 \cos 4x \cos x$.
10. $\cos (a + b) + \cos (a - b)$.
11. $\sin 3x - \sin x$. *Ans.* $2 \sin x \cos 2x$.
12. $\cos 4x - \cos 2x$.

Prove the following identities.

13. $\dfrac{\cos 3x - \cos x}{\cos 3x + \cos x} = -\tan 2x \tan x$.
14. $\dfrac{\sin 5x - \sin 3x}{\sin 5x + \sin 3x} = \tan x \cot 4x$.
15. $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x$.
16. $\dfrac{\cos 3x + \cos x}{\sin 3x + \sin x} = \cot 2x$.
17. $\dfrac{\cos 3x - \cos x}{\sin 3x - \sin x} = \dfrac{2 \tan x}{\tan^2 x - 1}$.
18. $\sin (45° + x) - \sin (45° - x) = \sqrt{2} \sin x$.

19. $\cos(45° + x) - \cos(45° - x) = -\sqrt{2}\sin x.$

20. $\sin\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y) + \cos\frac{1}{2}(x + y)\sin\frac{1}{2}(x - y) = \sin x.$

21. $\sin 5x + \sin 3x = 8\sin x\cos^2 x\cos 2x.$

22. $\cos 6x - \cos 2x = -16\sin^2 x\cos^2 x(2\cos^2 x - 1).$

23. $\cot 4x + \tan x = \dfrac{2\cos 3x}{\sin 5x + \sin 3x}.$

24. $1 + \tan 2x\cot 3x = \dfrac{2\sin 5x}{\sin x + \sin 5x}.$

25. $\cot x + \cot 2x = \dfrac{2\sin 3x}{\cos x - \cos 3x}.$

26. $1 + \tan 3x\tan 4x = \dfrac{2\cos x}{\cos 7x + \cos x}.$

27. $\dfrac{\cos 2A + \cos A + 1}{\sin 2A + \sin A} = \cot A.$

28. $\dfrac{\sin A + \cos 2A - 1}{\cos A - \sin 2A} = \tan A.$

29. $\dfrac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \tan\dfrac{x}{2}.$

30. $\sin 2x + \sin 4x + \sin 6x = 4\cos x\cos 2x\sin 3x.$

31. $\cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x - 1.$

32. $\dfrac{\cos 2x + \cos 7x}{\cos 3x + \cos 6x} + \dfrac{\cos 6x - \cos x}{\cos 5x - \cos 2x} = \dfrac{2\sin 4x}{\sin 3x}.$

33. $\dfrac{\sin 10x + \sin 6x}{\sin 12x + \sin 4x} - \dfrac{\sin 5x - \sin x}{\sin 7x + \sin x} = \dfrac{2\sin 2x}{\sin 8x}.$

34. $\dfrac{\cos 6x + \cos 12x}{\sin 14x - \sin 4x} + \dfrac{\sin 7x - \sin x}{\cos 9x + \cos x} = \dfrac{2\cos 2x}{\sin 10x}.$

35. $\dfrac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \tan 2x.$

36. $\sin^2 3x - \sin^2 x = 2\sin^2 2x\cos 2x.$

37. $\dfrac{\sin(x + 3y) + \sin(3x + y)}{\sin 2x + \sin 2y} = 2\cos(x + y).$

38. $\dfrac{\cos(x + 3y) - \cos(3x + y)}{\cos 2x - \cos 2y} = -2\cos(x + y).$

Solve the following equations for all values of x from 0° to 360°.

39. $\sin 2x = \sin x$. *Ans.* 0°, 60°, 180°, 300°, 360°.

40. $\sin 2x = \cos x$.

41. $\cos 2x = \cos x$. *Ans.* 0°, 120°, 240°, 360°.

42. $\cos 2x = \sin x$.

43. $\tan 2x + \tan x = 0$. *Ans.* 0°, 60°, 120°, 180°, 240°, 300°, 360°.

44. $\tan 2x = \tan x$.

45. $\sin 2x + \sin x = 0$. *Ans.* 0°, 120°, 180°, 240°, 360°.

46. $\sin 2x + \cos x = 0$.

47. $\cos 2x + \cos x = 0$. *Ans.* 60°, 180°, 300°.

48. $\cos 2x + \sin x = 0$.

49. $\cos 4x + \sin 2x = 0$. *Ans.* 45°, 105°, 165°, 225°, 285°, 345°.

50. $\sin 4x + \sin 2x = 0$.

51. $\sin \frac{x}{2} + \cos x = 1$. *Ans.* 0°, 60°, 300°, 360°.

52. $\cos \frac{x}{2} - \cos x = 1$.

53. $\tan \frac{x}{2} + \cos x = 1$. *Ans.* 0°, 90°, 360°.

54. $\cot \frac{x}{2} - \cos x = 1$.

55. $\sin \frac{x}{2} - \cos x = 1$. *Ans.* 102° 39.8′, 257° 20.2′.

56. $2 \tan x = 2 - \sec^2 x$.

57. $\sin 3x + \sin x = \cos x$. *Ans.* 15°, 75°, 90°, 195°, 255°, 270°.

58. $\cos 3x - \cos x = \sin x$.

59. $2 \sin 3x + 2 \sin x = \tan x$. *Ans.* 0°, 60°, 180°, 300°, 360°.

60. $\sin 3x + \sin x = \cot x \cos x$.

61. $\tan x - \cot x = \cot 2x$. *Ans.* 45°, 135°, 225°, 315°.

62. $(2 \cos 2x + 1)(\sin 3x - 3) = 0$.

63. $\tan 3x - \tan x = \sec 3x$. *Ans.* No solutions.

64. $\sin 5x + \sin x = \cos 2x$.

65. $\sin 5x - \sin x + \cos 4x + \cos 2x = 0$.

Ans. 30°, 90°, 150°, 210°, 270°, 330°.

66. $\cos 5x - \cos 3x + 2 \sin x = 0.$

67. $\sin 5x + 2 \cos x + \sin 3x = 0.$

Ans. $67\frac{1}{2}°, 90°, 157\frac{1}{2}°, 247\frac{1}{2}°, 270°, 337\frac{1}{2}°.$

68. $\cos 5x - \cos 3x + \sin x = 0.$

69. $\tan 3x + \tan x = 0.$ *Ans.* $0°, 45°, 135°, 180°, 225°, 315°, 360°.$

70. $\sin x + \sin 3x + \sin 5x = 0.$

71. $\cos x + \cos 2x + \cos 3x = 0.$

Ans. $45°, 120°, 135°, 225°, 240°, 315°.$

72. $\cos x + \cos 3x + \cos 5x = 0.$

Prove the following relations on the assumption that A and B are complementary angles.

73. $\sin (A - B) = \cos 2B.$

74. $\cos (A - B) = \sin 2A = \sin 2B.$

75. $\tan (A - B) = \cot 2B.$

Prove the following relations on the assumption that $A + B + C = 180°$.

76. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$

[*Solution:*

$$(\sin A + \sin B) + \sin C = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin C$$

$$= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2}.$$

Since $\frac{A+B}{2} + \frac{C}{2} = 90°$, $\sin \frac{A+B}{2} = \cos \frac{C}{2}$. Therefore

$$\sin A + \sin B + \sin C = 2 \cos \frac{C}{2} \left(\cos \frac{A-B}{2} + \sin \frac{C}{2} \right)$$

$$= 2 \cos \frac{C}{2} \left(\cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right)$$

$$= 2 \cos \frac{C}{2} \left(2 \cos \frac{A}{2} \cos \frac{B}{2} \right) = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.]$$

77. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

78. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
[*Suggestion:* $\tan C = -\tan (A + B)$.]

79. $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$.

Prove the following identities.

80. $\cos \alpha \tan (\theta - \alpha) + \sin \alpha = \dfrac{\sin \theta}{\cos (\theta - \alpha)}$.

81. $\dfrac{\sec^2\left(ax - \frac{\pi}{4}\right)}{\tan\left(ax - \frac{\pi}{4}\right)} = -2 \sec 2ax$.

82. $\sin^2 4x \cos^4 3x \cos 4x - \sin^3 4x \cos^3 3x \sin 3x = \sin^2 4x \cos^3 3x \cos 7x$.

83. $\csc^3 x + \cot^2 x \csc x + \dfrac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2}} = 2 \csc^3 x$.

84. $\dfrac{\cos \frac{1}{2}(\theta-\alpha) \sin \frac{1}{2}(\theta+\alpha) - \cos \frac{1}{2}(\theta+\alpha) \sin \frac{1}{2}(\theta-\alpha)}{2 \sin \frac{1}{2}(\theta-\alpha) \sin \frac{1}{2}(\theta+\alpha)} = \dfrac{\sin \alpha}{\cos \alpha - \cos \theta}$.

85. $\cos\left(2x - \frac{\pi}{4}\right)\sin\left(2x + \frac{\pi}{4}\right) - \cos\left(2x + \frac{\pi}{4}\right)\sin\left(2x - \frac{\pi}{4}\right) = 1$.

86. $\dfrac{\sec^2 \frac{x}{2}}{2\left(\tan \frac{x}{2} - 2\right)\left(2 \tan \frac{x}{2} - 1\right)} = \dfrac{1}{4 - 5 \sin x}$.

87. $\sin^4 x = \frac{1}{4}\left[1 - 2 \cos 2x + \dfrac{1 + \cos 4x}{2}\right]$.

88. $\cos^4 x = \frac{1}{4}\left[1 + 2 \cos 2x + \dfrac{1 + \cos 4x}{2}\right]$.

89. $\sin^2 x \cos^2 x = \frac{1}{8}(1 - \cos 4x)$.

90. $\sin^4 x \cos^4 x = \frac{1}{64}\left(1 - 2 \cos 4x + \dfrac{1 + \cos 8x}{2}\right)$.

CHAPTER IX

THE OBLIQUE TRIANGLE

80. The parts of a triangle. — The six *parts* of a triangle are its three sides and three angles. We shall commonly denote the vertices as A, B, C, the angles at the vertices as α, β, γ and the respectively opposite sides as a, b, c. If three parts are given, at least one of which is a side, it is possible to determine the other three parts except in one ambiguous case which will be described later. This determination of the unknown parts is the *solution* of the triangle.

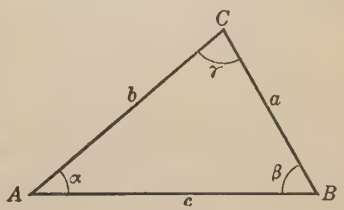

FIG. 55

In the paragraphs that follow we obtain certain laws or relations that exist among the parts of any triangle. One of these laws is already familiar to us. It is that *the sum of the angles of any triangle is equal to* 180°.

$$\alpha + \beta + \gamma = 180°.$$

This enables us to find one angle of a triangle if the other two angles are known.

The method of solution of a triangle depends on which set of three parts is given. The possible cases are as follows. We may be given:

I. One side and two angles.
II. Two sides and the angle opposite one of them.
III. Two sides and the included angle.
IV. Three sides.

81. The law of cosines. — The law of cosines is as follows: *In any triangle the square of any side is equal to the sum of the squares of the*

other two sides minus twice the product of those sides times the cosine of the included angle. As an equation this takes one of the forms

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$
$$b^2 = c^2 + a^2 - 2ca \cos \beta.$$
$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

Proof. In any given triangle, let a be any side. From an ad-

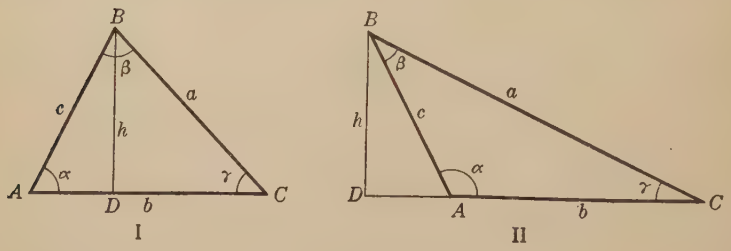

Fig. 56

jacent vertex, B, drop the perpendicular BD to AC or to AC produced. If α is acute, the figure is like Figure 56 I. If α is obtuse, the figure is like II.* Let h denote the length of $\overline{BD}$.

In either figure, in the right triangle CDB,

$$a^2 = h^2 + \overline{CD}^2.$$

In I, $\overline{DC} = b - \overline{AD} = b - c \cos \alpha,$

since $\overline{AD} = c \cos \alpha.$

In II, $\overline{DC} = b + \overline{DA} = b - c \cos \alpha,$

since $\overline{DA} = -c \cos \alpha.$

In either figure, $h = c \sin \alpha$. Therefore

$$\begin{aligned} a^2 &= c^2 \sin^2 \alpha + (b - c \cos \alpha)^2 \\ &= c^2 \sin^2 \alpha + b^2 - 2bc \cos \alpha + c^2 \cos^2 \alpha \\ &= b^2 + c^2 (\sin^2 \alpha + \cos^2 \alpha) - 2bc \cos \alpha, \end{aligned}$$

* If α is acute and γ is obtuse, D falls on AC produced, but the equations are the same as for Figure I. If α is a right angle, $\cos \alpha = 0$, and the law of cosines becomes $a^2 = b^2 + c^2$, which is known to be true since a, b, and c are the sides of a right triangle.

which gives the result

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

This proves the law of cosines for all sides, since a was any side.

82. Applications of the law of cosines. — Besides its use in the solution of triangles, the law of cosines has many applications in other fields of geometry — more applications probably than the other laws which we are to develop. Its use in solving triangles is limited, however, because it does not lend itself to logarithmic computation. For, since the three terms are to be added, it is impossible to combine them by means of logarithms. In examples in which the numbers are small and easy to handle without logarithms and when two sides are given together with the included angle, the law of cosines may be used to advantage in finding the third side. And if three sides are given, the law of cosines may be used to find the angles.

EXAMPLE 1. Given $b = 4$, $c = 3$, $\alpha = 60°$, find a. From the formula,

$$a^2 = 4^2 + 3^2 - 2 \times 4 \times 3 \times \tfrac{1}{2} = 13.$$

$$a = \sqrt{13} = 3.6.$$

EXAMPLE 2. Given $a = 6$, $b = 5$, $c = 4$, find β. We use the law of cosines in the form

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

or

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{36 + 16 - 25}{2 \times 6 \times 4} = \frac{9}{16} = .56.$$

From a table of cosines, we find $\beta = 56°$ to the nearest degree.

EXERCISES

1. Given $a = 5$, $c = 3$, $\beta = 45°$, find b. *Ans.* 3.6.
2. Given $a = 7$, $b = 5$, $\gamma = 30°$, find c.
3. Given $a = 4$, $b = 2$, $c = 3$, find α. *Ans.* $104\frac{1}{2}°$.
4. Given $a = 5$, $b = 3$, $c = 6$, find β.
5. Given $b = 12$, $c = 8$, $\alpha = 44°$, find a. *Ans.* 8.4.

6. Given $a = 7$, $c = 9$, $\beta = 61°$, find b.

7. Given $b = 10$, $c = 15$, $\alpha = 112°$, find a. *Ans.* 20.9.

8. Given $a = 8$, $b = 6$, $\gamma = 138°$, find c.

9. An observer is 200 feet from one end of a pond and 300 feet from the other. He observes the angle subtended by the pond and finds it to be 120°. Find the length of the pond. *Ans.* 436 ft.

10. A body is acted upon by a force of 90 lbs. and by another force of 75 lbs. which makes an angle of 55° with the first force. Find the magnitude of the resultant force.

11. Sound travels at the rate of about 1100 feet per second. From a hill near a lake, the angle subtended by the line joining two boats on the lake is 15°. A pistol shot fired at the top of the hill is heard 1 second later in the nearer boat and $1\frac{2}{5}$ seconds after firing in the further boat. How far apart are the boats? *Ans.* 753 ft.

83. The law of sines. — *In any triangle the sides are proportional to the sines of the respectively opposite angles.*

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

The law may be proved in the following manner.

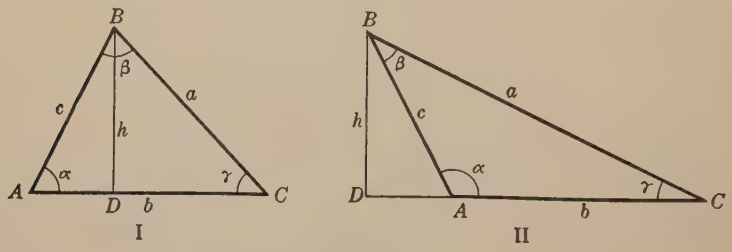

Fig. 57

Given any triangle, ABC, draw the figure just as in Article **81**. There are two cases, one in which α and γ are both acute angles, as in I; the other in which either α or γ is obtuse. In this case, for definiteness, suppose that α is the obtuse angle, as in II.

In either figure,

$$h = a \sin \gamma,$$

as we may see from the right triangle DCB.

From the right triangle ADB we find that $h = c \sin \angle DAB$. In I, $\angle DAB = \alpha$. In II, $\angle DAB$ is the supplement of α. In either case, $\sin \angle DAB = \sin \alpha$. Therefore

$$h = c \sin \alpha.$$

Equating the two values of h, we have

$$a \sin \gamma = c \sin \alpha.$$

Therefore

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}.$$

This shows that any two sides are proportional to the sines of the opposite angles, so that

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

84. Applications of the law of sines. — The law of sines lends itself readily to logarithmic work, since it involves only ratios and products. It may be used in solving a triangle whenever the given parts include a side and the opposite angle. Moreover the law of sines may be used in completing or in checking solutions undertaken by other laws. In general, any solution should be checked by some method involving the use of other formulas than those used in the original solution, and, of course, all of the parts not given. Until we have obtained other formulas we shall postpone logarithmic checks.

There are two cases of triangle problems in which the given parts include a side and the opposite angle, Case I of Article **80**, which we will now consider, and Case II, which will be discussed in the next article.

CASE I. *Given one side and two angles.* — Knowing two angles, we can find the third angle from the relation

$$\alpha + \beta + \gamma = 180°.$$

Suppose, for definiteness, that we are given side a; it remains to find b and c. This we can do since, by the law of sines,

$$b = \frac{a}{\sin \alpha} \cdot \sin \beta \quad \text{and} \quad c = \frac{a}{\sin \alpha} \cdot \sin \gamma.$$

Since $a/\sin \alpha$ appears in each expression, in solving by logarithms, one first obtains

$$\log (a/\sin \alpha) = \log a - \log \sin \alpha$$

before adding $\log \sin \beta$ or $\log \sin \gamma$.

Before beginning the solution it is always well to make a sketch approximately to scale.

EXAMPLE 1. Solve the triangle, given $\beta = 30°$, $\gamma = 45°$, $b = 6$.

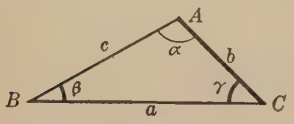

FIG. 58

Evidently

$$\alpha = 180° - (30° + 45°) = 105°.$$

By the law of sines

$$c = \frac{b}{\sin \beta} \cdot \sin \gamma = \frac{6}{\frac{1}{2}} \times \tfrac{1}{2}\sqrt{2} = 6\sqrt{2} = 8.5.$$

$$a = \frac{b}{\sin \beta} \cdot \sin \alpha = \frac{6}{\frac{1}{2}} \times .97 = 11.6.$$

EXAMPLE 2. Solve the triangle, given $a = 54.87$, $\beta = 43° 26'$, $\gamma = 58° 19'$.

We first find

$$\alpha = 180° - (\beta + \gamma) = 180° - 101° 45' = 78° 15'.$$

By the law of sines

$$b = \frac{a}{\sin \alpha} \cdot \sin \beta = \frac{54.87}{\sin 78° 15'} \cdot \sin 43° 26',$$

$$c = \frac{54.87}{\sin 78° 15'} \cdot \sin 58° 19'.$$

We arrange the work as follows:

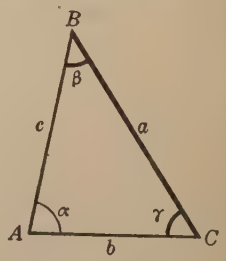

FIG. 59

$$\begin{aligned} \log 54.87 &= \\ \log \sin 78^\circ\, 15' &= \underline{\qquad\qquad} (-) \\ \log (a/\sin\alpha) &= \\ \log \sin 43^\circ\, 26' &= \underline{\qquad\qquad} (+) \\ \log b &= \\ b &= \end{aligned} \qquad \begin{aligned} \log (a/\sin\alpha) &= \\ \log \sin 58^\circ\, 19' &= \underline{\qquad\qquad} (+) \\ \log c &= \\ c &= \end{aligned}$$

On filling in the blank spaces, we get

$$\begin{aligned} \log 54.87 &= 1.73933 \\ \log \sin 78^\circ\, 15' &= 9.99080 - 10\ (-) \\ \log (a/\sin\alpha) &= 1.74853 \\ \log \sin 43^\circ\, 26' &= 9.83728 - 10\ (+) \\ \log b &= 1.58581 \\ b &= 38.53 \end{aligned} \qquad \begin{aligned} \log (a/\sin\alpha) &= 1.74853 \\ \log \sin 58^\circ\, 19' &= 9.92991 - 10\ (+) \\ \log c &= 1.67844 \\ c &= 47.69 \end{aligned}$$

EXAMPLE 3. Solve the triangle ABC, given $b = 3$, $c = 2$, $\alpha = 60^\circ$. This is Case III, two sides and the included angle being given. By the law of cosines

$$a^2 = 9 + 4 - 2 \times 3 \times 2 \times \tfrac{1}{2} = 7,$$
$$a = \sqrt{7} = 2.65.$$

By the law of sines

$$\sin\beta = \frac{\sin\alpha}{a} \cdot b = \frac{.866}{2.65} \times 3 = .980$$

and

$$\sin\gamma = \frac{\sin\alpha}{a} \cdot c = .653.$$

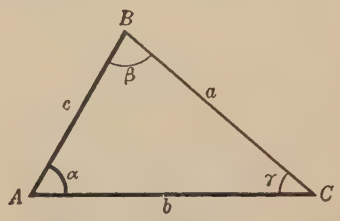

FIG. 60

Therefore

$$\beta = 79^\circ, \quad \gamma = 41^\circ \text{ to the nearest degree.}$$

We can check by the law of cosines, writing

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} = \frac{7 + 4 - 9}{2\,(2.65)\,2} = .189, \qquad \beta = 79^\circ,$$

and

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} = \frac{7 + 9 - 4}{2\,(2.65)\,3} = .754, \qquad \gamma = 41^\circ.$$

EXERCISES

Solve the following triangles without logarithms.

1. Given $a = 10$, $\beta = 45°$, $\gamma = 60°$.

Ans. $\alpha = 75°$, $b = 7.3$, $c = 8.9$.

2. Given $a = 8$, $\beta = 30°$, $\gamma = 56°$.

3. Given $b = 5$, $\beta = 42°$, $\gamma = 28°$.

Ans. $\alpha = 110°$, $a = 7.0$, $c = 3.5$.

4. Given $c = 11$, $\alpha = 39°$, $\gamma = 53°$.

5. Given $a = 4$, $b = 6$, $\gamma = 60°$.

Ans. $c = 5.3$, $\alpha = 40° 50'$, $\beta = 79° 10'$.

6. Given $b = 20$, $c = 30$, $\alpha = 40°$.

Solve the following triangles, using logarithms.

7. Given $b = 27.36$, $\beta = 24° 18'$, $\gamma = 32° 24'$.

Ans. $\alpha = 123° 18'$, $a = 55.57$, $c = 35.63$.

8. Given $c = 541.7$, $\alpha = 62° 43'$, $\gamma = 48° 17'$.

9. Given $a = 6425$, $\beta = 73° 29'$, $\gamma = 36° 52'$.

Ans. $\alpha = 69° 39'$, $b = 6570$, $c = 4111$.

10. Given $a = 102.3$, $\alpha = 16° 16'$, $\gamma = 38° 42'$.

11. Given $c = 14.327$, $\beta = 68° 22.7'$, $\gamma = 29° 26.3'$.

Ans. $\alpha = 82° 11.0'$, $b = 27.100$, $a = 28.880$.

12. Given $b = 624.38$, $\alpha = 79° 23.5'$, $\beta = 33° 44.8'$.

13. Given $a = 4327.4$, $\alpha = 36° 16.8'$, $\beta = 42° 24.6'$.

Ans. $\gamma = 101° 18.6'$, $b = 4932.1$, $c = 7171.0$.

14. Given $c = 602.49$, $\beta = 43° 16'$, $\gamma = 111° 18'$.

15. Two points, A and B, are on opposite sides of a stream. In order to find AB, a point C is taken on the same side of the stream as A and 1000.0 feet from A. By measurement $\angle CAB = 33° 27'$, $\angle ACB = 47° 32'$. Find AB. *Ans.* 746.9 ft.

16. A body is acted upon by a force of 973 lbs. and by a force of unknown magnitude which makes an angle of $49° 10'$ with the direction of the first force. The resultant force can be measured and is found to be 1268 lbs. Find the magnitude of the second force and the angle which it makes with the resultant force.

85. The ambiguous case. — Case II, in which we are given two sides and the angle opposite one of them, is called the *ambiguous case*, because the conditions are sometimes not sufficient to determine the triangle uniquely. Suppose, for definiteness, that we are given the sides a and b and the angle α. From these data we attempt to draw the triangle. We first draw $\overline{AC} = b$. From AC we lay off the angle α getting the direction AD in which the side c is to lie. Not knowing c, we cannot yet fix the point B on AD. With C as center we now draw an arc of a circle of radius a. The third vertex, B, can be taken where this arc meets the line AD.

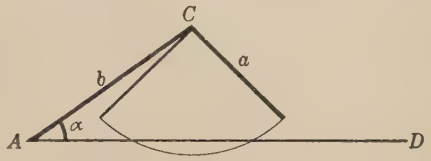

FIG. 61

In case the given a is less than b, there are three possibilities:

(i) If a is just equal to the perpendicular line from C to AD, the point B is at the foot of this perpendicular, and can occupy no other position. The triangle is a right triangle, and $\sin \beta = 1$. By the law of sines

$$\sin \beta = \frac{b \sin \alpha}{a},$$

so that this case occurs whenever $b \sin \alpha / a = 1$.

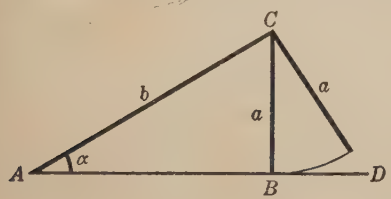

FIG. 62

(ii) If a is longer than the perpendicular line from C to AD, the arc intersects the line AD in two points, B and B', each of which is the vertex of a triangle which satisfies the given conditions. Because the triangle is not determined uniquely,

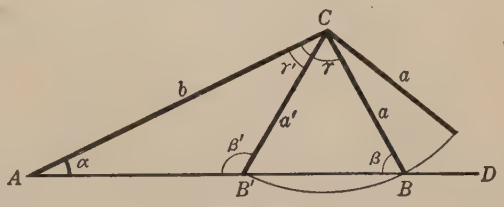

FIG. 63

this is referred to as the ambiguous case, in the strict sense of the term, though the same name is given to the whole group of cases in which two sides and the angle opposite one of them are given. For a complete solution, it is necessary to solve both triangles ABC and $AB'C$, though in a practical problem it is usually evident with which triangle we must deal.

In this case a is longer, for a given b and α, than in the preceding case, so that $b \sin \alpha / a$ is less than 1. This, of course, was to be expected since, when β is not a right angle,

$$\sin \beta = \frac{b \sin \alpha}{a} \text{ is less than 1.}$$

(iii) The third case occurs when a is shorter than the perpendicular line from C to AD. In this case no triangle exists satisfying the given conditions. For a given b and α, a is smaller than for the case (i), so that $b \sin \alpha / a$ is greater than 1. This indicates the impossibility of satisfying the given conditions, since for no value of β is $\sin \beta$ greater than 1.

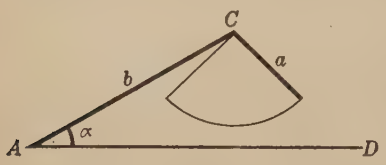

Fig. 64

(iv) There remains the case in which a is greater than or equal to b. In this case the triangle always exists and is uniquely determined. For the arc cuts the line AD in two points B and B' which lie on opposite sides of A, if a is greater than b. And, of the two triangles ABC and $AB'C$, one contains the given angle α; the other contains the supplement of α but not α itself. If a is just equal to b, B' falls at A, and only one triangle is possible. The triangle in this case is isosceles.

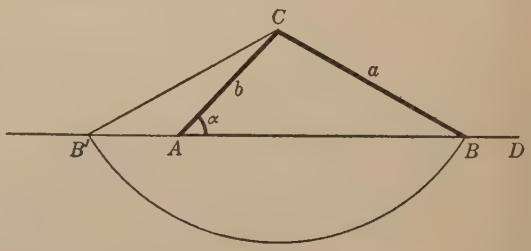

Fig. 65

We can say then that:

(i) if a is less than b, and if $b \sin \alpha / a = 1$, the triangle exists, is unique, and is a right triangle;

(ii) if a is less than b, and if $b \sin \alpha / a$ is less than 1, there exist two triangles satisfying the conditions.

(iii) if a is less than b, and if $b \sin \alpha / a$ is greater than 1, no triangle exists satisfying the conditions.

(iv) if a is greater than or equal to b, just one triangle exists satisfying the given conditions.

In the solution by logarithms, when a is less than b,

(i) if $\log (b \sin \alpha / a) = 0$, one right triangle exists;

(ii) if $\log (b \sin \alpha / a)$ is negative, two triangles exist;

(iii) if $\log (b \sin \alpha / a)$ is positive, no triangle exists.

EXAMPLE 1. Solve the triangle ABC, given

$$a = 6, \quad b = 8, \quad \alpha = 30°.$$

By the law of sines

$$\sin \beta = \frac{\sin \alpha}{a} \cdot b$$

$$= \frac{1}{2 \times 6} \times 8 = \frac{2}{3} = .667,$$

which gives

$$\beta = 42°,$$

$$\beta' = 180° - 42° = 138°.$$

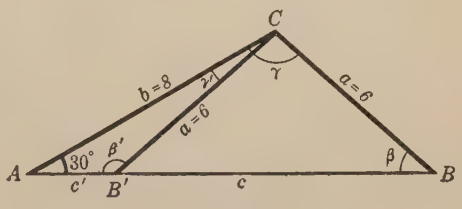

FIG. 66

Then

$$\gamma = 180° - (30° + 42°) = 108°, \gamma' = 180° - (30° + 138°) = 12°$$

and

$$c = \frac{a}{\sin \alpha} \cdot \sin \gamma = 2 \times 6 \times .95 = 11.4,$$

$$c' = \frac{a}{\sin \alpha} \cdot \sin \gamma' = 2 \times 6 \times .21 = 2.5.$$

EXAMPLE 2. Solve the triangle ABC, given

$$a = 4, \quad b = 8, \quad \alpha = 30°.$$

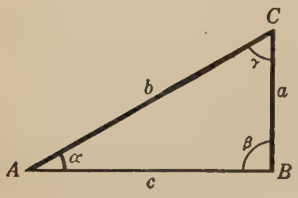

FIG. 67

By the law of sines,

$$\sin \beta = \frac{\sin \alpha}{a} \cdot b = \frac{1}{2 \times 4} \times 8 = 1,$$

so that $\beta = 90°$,

and the triangle is unique. Evidently

$$\gamma = 60°,$$

and

$$c = b \cos \alpha = .866 \times 8 = 6.9.$$

EXAMPLE 3. Solve the triangle ABC, given

$$a = 3, \quad b = 8, \quad \alpha = 30°.$$

By the law of sines

$$\sin \beta = \frac{\sin \alpha}{a} \cdot b = \frac{1}{2 \times 3} \times 8 = \frac{4}{3}.$$

Since this is greater than 1, the triangle is impossible.

EXAMPLE 4. Solve the triangle ABC, given

$$a = 21.34,$$
$$b = 27.56,$$
$$\alpha = 34° 26',$$
$$\sin \beta = \frac{\sin \alpha}{a} \cdot b.$$

$$\log \sin \alpha = 9.75239 - 10$$
$$\log a = \underline{1.32919 \quad (-)}$$
$$\log\left(\frac{\sin \alpha}{a}\right) = 8.42320 - 10$$
$$\log b = \underline{1.44028 \quad (+)}$$
$$\log \sin \beta = 9.86348 - 10$$

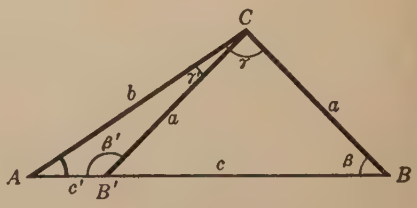

FIG. 68

Since this logarithm is negative, and since a is less than b, we see that two triangles exist satisfying the given conditions. Call them ABC and $AB'C'$.

In the triangle ABC, from the value of $\log \sin \beta$, we find $\beta = 46° 54'$.

$$\gamma = 180° - (\alpha + \beta) = 180° - 81° 20' = 98° 40'.$$

$$c = \frac{a}{\sin \alpha} \sin \gamma.$$

$$\log \sin \gamma = \log \sin 81° 20' = 9.99501 - 10$$

$$\log \left(\frac{\sin \alpha}{a} \right) = \underline{8.42320 - 10} \quad (-)$$

$$\log c = 1.57181$$

$$c = 37.31.$$

For the triangle $AB'C'$, since $B'CB$ is isosceles,

$$\beta' = 180° - \beta = 133° 6'.$$

$$\gamma' = 180° - (\alpha + \beta') = 12° 28'.$$

$$\log \sin \gamma' = 9.33420 - 10$$

$$\log \left(\frac{\sin \alpha}{a} \right) = \underline{8.42320 - 10} \quad (-)$$

$$\log c' = 0.91100$$

$$c' = 8.15.$$

EXAMPLE 5. Solve the triangle ABC, given

$$a = 5.632$$
$$b = 7.254$$
$$\alpha = 64° 18'.$$

As in the preceding example, we seek

$$\sin \beta = \frac{\sin \alpha}{a} \cdot b$$

$$\log \sin \alpha = 9.95476 - 10$$

$$\log a = \underline{0.75066} \quad (-)$$

$$\log \left(\frac{\sin \alpha}{a} \right) = 9.20410 - 10$$

$$\log b = \underline{0.86058} \quad (+)$$

$$0.06468$$

Since this logarithm is positive, no triangle exists which satisfies the given conditions, and we need proceed no further.

EXAMPLE 6. Solve the triangle ABC, given

$$a = 97.636$$
$$c = 64.680$$
$$\gamma = 41° 29.3'.$$

We use the relation

$$\sin \alpha = \frac{\sin \gamma}{c} \cdot a.$$

$$\begin{aligned} \log \sin \gamma &= 9.82116 - 10 \\ \log c &= 1.81077 \qquad (-) \\ &\overline{8.01039 - 10} \\ \log a &= 1.98961 \\ \log \sin \alpha &= \overline{10.00000 - 10.} \end{aligned}$$

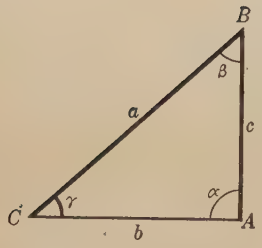

FIG. 69

Therefore, within the limits of accuracy of observation, $\alpha = 90°$, and the triangle is unique.

$$\begin{aligned} \beta &= 90° - \gamma = 48° 30.7'. \\ b &= a \cos \gamma. \\ \log a &= 1.98961 \\ \log \cos \gamma &= 9.87454 - 10 \quad (+) \\ \log b &= 1.86415 \\ b &= 73.140. \end{aligned}$$

EXERCISES

Find the number of possible triangles for each of the following sets of data, and complete the solution of all possible triangles. Solve Examples **1** to **4** without logarithms.

1. $a = 8, \alpha = 34°, b = 12.$

Ans. $\beta = 57°, \gamma = 89°, c = 14.3;$
$\beta' = 123°, \gamma' = 23°, c' = 5.6.$

2. $c = 10, \gamma = 49°, a = 12.$

3. $b = 9, \beta = 24°, c = 9.$ *Ans.* $\gamma = 24°, \alpha = 132°, a = 16.4.$

4. $a = 21, \alpha = 35°, c = 21.$

5. $a = 65.6, \alpha = 64°, c = 73.0.$

Ans. $\gamma = 90°, \beta = 26°, b = 32.0.$

6. $b = 96.0, c = 46.5, \gamma = 29°.$

7. $a = 12, b = 14, \beta = 38°.$

Ans. $\alpha = 31° 50', \gamma = 110° 10', c = 21.3.$

8. $b = 13, c = 20, \gamma = 53°.$

9. $b = 68.479, \gamma = 72° 12.3', c = 41.623.$ *Ans.* Impossible.

10. $a = 246.35$, $\alpha = 65° 46.7'$, $c = 289.58$.

11. $b = 7073.2$, $\beta = 59° 27.8'$, $c = 7836.5$.

Ans. $\gamma = 72° 36.2'$, $\alpha = 47° 56.0'$, $a = 6096.4$;
$\gamma' = 107° 23.8'$, $\alpha' = 13° 8.4'$, $a' = 1866.9$.

12. $a = 918.29$, $b = 746.63$, $\beta = 32° 19.4'$.

13. $a = 976.34$, $b = 602.57$, $\alpha = 109° 44.6'$.

Ans. $\beta = 35° 30.8'$, $\gamma = 34° 44.6'$, $c = 591.17$.

14. $a = 421.83$, $c = 617.68$, $\gamma = 131° 34.7'$.

15. In a children's playground a toboggan slide is 28 feet long and makes an angle of 39° with the ground. Its top is reached by a ladder 18 feet long. How steep is the ladder? *Ans.* 78°.

16. In a studio the north side of the roof is of glass. It is 18 feet from eaves to ridge and makes an angle of 68° with the horizontal. The south side of the roof is 31 feet from eaves to ridge. How wide is the studio?

86. The law of tangents. — *In any triangle the difference between two sides is to their sum as the tangent of one-half of the difference between the opposite angles is to the tangent of one-half of the sum of those angles:*

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)},$$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(\beta - \gamma)}{\tan \frac{1}{2}(\beta + \gamma)},$$

$$\frac{c-a}{c+a} = \frac{\tan \frac{1}{2}(\gamma - \alpha)}{\tan \frac{1}{2}(\gamma + \alpha)}.$$

Proof. In the triangle ABC, let a and b be any two sides and α and β the respectively opposite angles. By the law of sines

$$\frac{a}{b} = \frac{\sin \alpha}{\sin \beta}.$$

Therefore

$$\frac{a}{b} - 1 = \frac{\sin \alpha}{\sin \beta} - 1, \quad \text{and} \quad \frac{a}{b} + 1 = \frac{\sin \alpha}{\sin \beta} + 1.$$

Consequently

$$\frac{a-b}{b} = \frac{\sin\alpha - \sin\beta}{\sin\beta},$$

and

$$\frac{a+b}{b} = \frac{\sin\alpha + \sin\beta}{\sin\beta}.$$

Dividing the first of these two equations by the second, we get

$$\frac{a-b}{a+b} = \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}.$$

Therefore, by formulas XX and XXI of Chapter VIII,

$$\frac{a-b}{a+b} = \frac{2\sin\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\alpha+\beta)}{2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)} = \frac{\dfrac{\sin\tfrac{1}{2}(\alpha-\beta)}{\cos\tfrac{1}{2}(\alpha-\beta)}}{\dfrac{\sin\tfrac{1}{2}(\alpha+\beta)}{\cos\tfrac{1}{2}(\alpha+\beta)}},$$

and

$$\frac{a-b}{a+b} = \frac{\tan\tfrac{1}{2}(\alpha-\beta)}{\tan\tfrac{1}{2}(\alpha+\beta)}.$$

Since a and b are any two sides, this establishes the law.

87. Applications of the law of tangents. — The law of tangents may be used to solve a triangle when two sides and the included angle are given. It is usually preferable to the law of cosines, for it is well adapted to logarithmic computation. Suppose that a and b and the included angle γ are given. If a is greater than b, we use the law of tangents in the form:

$$\tan\tfrac{1}{2}(\alpha-\beta) = \frac{a-b}{a+b}\tan\tfrac{1}{2}(\alpha+\beta).$$

Since a and b are given, $(a-b)$ and $(a+b)$ can be found. Since $\gamma = 180^\circ - (\alpha+\beta)$, $\tfrac{1}{2}(\alpha+\beta) = 90^\circ - \tfrac{1}{2}\gamma$ may be found. This

enables us to find $\frac{1}{2}(\alpha - \beta)$ by the law of tangents. Having $\frac{1}{2}(\alpha + \beta)$ and $\frac{1}{2}(\alpha - \beta)$, by addition we find α, and by subtraction we find β. We then solve for c by the law of sines, and check by the same law.

EXAMPLE 1. Solve the triangle ABC, given

$$a = 564.7$$
$$b = 387.2$$
$$\gamma = 48° 28'.$$

Solution:

$$a = 564.7$$
$$b = 387.2$$
$$a - b = 177.5$$
$$a + b = 951.9$$
$$\tfrac{1}{2}(\alpha + \beta) = 90° - \tfrac{1}{2}\gamma = 90° - 24° 14' = 65° 46'.$$

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(\alpha + \beta)$$

$$\log (a - b) = 12.24920 - 10$$
$$\log (a + b) = \underline{2.97859 \quad (-)}$$
$$9.27061 - 10$$
$$\log \tan \tfrac{1}{2}(\alpha + \beta) = \underline{0.34667 \quad (+)}$$
$$\log \tan \tfrac{1}{2}(\alpha - \beta) = 9.61728 - 10$$

Therefore $\tfrac{1}{2}(\alpha - \beta) = 22° 30'.$
But $\tfrac{1}{2}(\alpha + \beta) = 65° 46'.$
Adding, we get $\alpha = 88° 16'.$
Subtracting, $\beta = 43° 16'.$
To find c, we set

$$c = \frac{a}{\sin \alpha} \cdot \sin \gamma.$$

$$\log a = 12.75182 - 10$$
$$\log \sin \alpha = \underline{9.99980 - 10 \ (-)}$$
$$2.75202$$
$$\log \sin \gamma = \underline{9.87423 - 10 \ (+)}$$
$$\log c = 2.62625$$
$$c = 422.9.$$

Check:

$$c = \frac{b}{\sin \beta} \sin \gamma$$

$$\log b = 12.58794 - 10$$
$$\log \sin \beta = \underline{9.83594 - 10 \ (-)}$$
$$2.75200$$
$$\log \sin \gamma = \underline{9.87423 - 10 \ (+)}$$
$$2.62623$$
$$c = 422.9.$$

The results should check within five figures if carried to five digits, and within two figures if carried to four digits, unless the angles are very close to 90° or 0°, in which case, due to the slow variation of the functions, the check may not be so good.

EXAMPLE 2. Solve the triangle ABC, given

$$a = 68.46$$
$$\gamma = 91^\circ\, 38'$$
$$\beta = 54^\circ\, 46',$$

and check the solution.

We first write $\alpha = 180^\circ - (\beta + \gamma) = 33^\circ\, 36'$,

$$b = \frac{a}{\sin \alpha} \cdot \sin \beta, \qquad c = \frac{a}{\sin \alpha} \cdot \sin \gamma.$$

$$\begin{aligned} \log a &= 11.83544 - 10 \\ \log \sin \alpha &= \underline{9.74303 - 10\ (-)} \\ \log \frac{a}{\sin \alpha} &= 2.09241 \\ \log \sin \beta &= \underline{9.91212 - 10\ (+)} \\ \log b &= 2.00453 \\ b &= 101.05 \end{aligned}$$

$$\begin{aligned} \log \frac{a}{\sin \alpha} &= 2.09241 \\ \log \sin \gamma &= \underline{9.99982 - 10\ (+)} \\ \log c &= 2.09223 \\ c &= 123.66. \end{aligned}$$

Check:

$$\frac{\tan \tfrac{1}{2}(\gamma - \beta)}{\tan \tfrac{1}{2}(\gamma + \beta)} = \frac{c - b}{c + b}.$$

$$\begin{aligned} c - b &= 22.61 \\ c + b &= 224.71 \end{aligned}$$

$$\begin{aligned} \gamma + \beta &= 146^\circ\, 24', \ \tfrac{1}{2}(\gamma + \beta) = 73^\circ\, 12' \\ \gamma - \beta &= 36^\circ\, 52', \ \tfrac{1}{2}(\gamma - \beta) = 18^\circ\, 26' \end{aligned}$$

$$\begin{aligned} \log (c - b) &= 11.35430 - 10 \\ \log (c + b) &= \underline{2.35162\ (-)} \\ &\quad 9.00268 - 10 \end{aligned}$$

$$\begin{aligned} \log \tan 18^\circ\, 26' &= 9.52284 - 10 \\ \log \tan 73^\circ\, 12' &= \underline{0.52011\ (-)} \\ \text{checking with} \quad &\quad 9.00273 - 10 \end{aligned}$$

EXERCISES

Solve the following triangles, and check.

1. $a = 28$, $b = 34$, $\gamma = 49^\circ$.

Ans. $\alpha = 53^\circ\, 30'$, $\beta = 77^\circ\, 30'$, $c = 26.3$.

2. $b = 48$, $c = 39$, $\alpha = 84^\circ$.

3. $b = 879$, $c = 498$, $\alpha = 33^\circ\, 10'$.

Ans. $\beta = 116^\circ\, 20'$, $\gamma = 30^\circ\, 30'$, $a = 537$.

4. $a = 312$, $c = 1293$, $\beta = 124^\circ\, 40'$.

5. $a = 5968$, $c = 7034$, $\beta = 39^\circ\, 27'$.

Ans. $b = 4501$, $\alpha = 57^\circ\, 24'$, $\gamma = 83^\circ\, 9'$.

6. $a = 62.37$, $b = 81.42$, $\gamma = 59^\circ\, 38'$.

7. $a = 536.8$, $b = 1124.4$, $\gamma = 129° 24.6'$. *Ans.* $\alpha = 15° 48.4'$, $\beta = 34° 47.0'$, $c = 1522.8$.

8. $b = 691.4$, $c = 1263.8$, $\alpha = 104° 39.4'$.

9. $\alpha = 64° 26.3'$, $\beta = 38° 54.9'$, $a = 6246.7$. *Ans.* $\gamma = 76° 38.8'$, $b = 4349.6$, $c = 6737.0$.

10. $\beta = 42° 35.3'$, $\gamma = 29° 53.8'$, $a = 7983.4$.

11. $b = 32698$, $c = 28463$, $\alpha = 78° 13.8'$. *Ans.* $\beta = 55° 45.1'$, $\gamma = 46° 1.1'$, $a = 38725$.

12. $a = 5674.3$, $b = 7327.9$, $\gamma = 68° 49.3'$.

13. $b = 6842.3$, $c = 8967.7$, $\gamma = 62° 34.4'$. *Ans.* $\alpha = 74° 47.9'$, $\beta = 42° 37.7'$, $a = 9749.8$.

14. $a = 594.03$, $b = 521.74$, $\beta = 58° 16.7'$.

15. To find the distance between two points A and B at opposite ends of a lake, a third point is chosen 256.7 rods from one and 178.4 rods from the other. At this point the angle subtended by the line AB is $57° 32'$. Find the distance AB. *Ans.* 220.4 rods.

16. A body is subject to a force of 873.4 pounds and to a second force of 561.8 pounds acting in a direction at $71° 24'$ from the direction of the first force. Find the magnitude and direction of the resultant force.

88. Half-angle formulas. — When three sides of a triangle are given, the angles may be found by using the law of cosines. But this method is convenient only if the numbers are easily handled without logarithms. Starting with the law of cosines, we now obtain formulas in terms of the sides of the triangle for the functions of $\alpha/2$, $\beta/2$, $\gamma/2$, that is, of angles which are half as large as the angles of the triangle.

By formula XIII, Chapter VIII,

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}}. \tag{1}$$

If α is any one of the angles of a triangle, by the law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$

so that

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}.$$

Therefore

$$\sin^2 \frac{\alpha}{2} = \tfrac{1}{2}(1 - \cos \alpha) = \tfrac{1}{2}\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{a^2 - b^2 + 2bc - c^2}{4bc}$$

$$= \frac{a^2 - (b - c)^2}{4bc}.$$

We can factor this, getting

$$\sin^2 \frac{\alpha}{2} = \frac{(a - b + c)}{2} \cdot \frac{(a + b - c)}{2bc}. \tag{2}$$

The formulas become more symmetrical, if we denote by s *one-half* of the perimeter of the triangle, or

$$s = \frac{a + b + c}{2}. \tag{3}$$

This gives

$$s - a = \frac{a + b + c}{2} - a = \frac{b + c - a}{2},$$

and

$$s - a = \frac{b + c - a}{2}, \quad s - b = \frac{a - b + c}{2}, \quad s - c = \frac{a + b - c}{2}. \tag{4}$$

Substituting in equation (2), we obtain

$$\sin^2 \frac{\alpha}{2} = \frac{(s - b)(s - c)}{bc}.$$

Finally

$$\sin \frac{\alpha}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

Likewise

$$\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \qquad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

In a similar way, we write

$$\begin{aligned}\cos^2\frac{\alpha}{2} = \tfrac{1}{2}(1+\cos\alpha) &= \tfrac{1}{2}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\\ &= \frac{(b^2+2bc+c^2)-a^2}{4bc}\\ &= \frac{(b+c-a)}{2}\cdot\frac{(a+b+c)}{2bc}\\ &= \frac{s(s-a)}{bc}.\end{aligned}$$

Therefore

$$\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}, \quad \cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

In all cases the radicals are positive since they represent sines and cosines of acute angles.

89. The tangent of the half-angle. — Division of $\sin(\alpha/2)$ by $\cos(\alpha/2)$ gives

$$(1) \qquad \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}},$$

or

$$(2) \qquad \tan\frac{\alpha}{2} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

To simplify this expression, set

$$(3) \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

In Article **91** it will be shown that r is equal to the radius of the circle which can be inscribed in the triangle. For the present we need only

think of r as a symbol for the expression in equation (3). We then get

$$(4) \qquad \tan\frac{\alpha}{2} = \frac{r}{s-a}; \quad \tan\frac{\beta}{2} = \frac{r}{s-b}; \quad \tan\frac{\gamma}{2} = \frac{r}{s-c}.$$

90. Applications of the half-angle formulas. — These formulas are used for finding the angles of a triangle when all of the sides are given. If only one of the angles is desired, the sine or cosine formula is satisfactory unless the angle is very large or very small. The tangent formulas are suitable in all cases. And if one wishes to find all of the angles it is always best to use the tangent formulas, since $\log r$ may be computed once for all. The method also furnishes a convenient check for other solutions.

EXAMPLE 1. Solve the triangle ABC, given

$$\begin{aligned} a &= 49.63 \\ b &= 57.98 \\ c &= 36.11 \\ \text{Adding,} \quad 2s &= 143.72. \end{aligned}$$

$$s = 71.86 \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\begin{aligned} s - a &= 22.23 \\ s - b &= 13.88 \\ s - c &= 35.75 \\ \text{Add } 3s - (a+b+c) &= 71.86 = s \end{aligned}$$

(This addition gives a check, since we should have $3s - (a+b+c) = 3s - 2s = s$.)

$$\begin{aligned} \log(s-a) &= 1.34694 \\ \log(s-b) &= 1.14239 \\ \log(s-c) &= 1.55328 \quad (+) \\ & \quad 4.04261 \\ \log s &= 1.85649 \quad (-) \\ \log r^2 &= 2.18612 \quad (\div 2) \\ \log r &= 1.09306 \end{aligned}$$

$$\tan\frac{\alpha}{2} = \frac{r}{s-a}, \qquad \tan\frac{\beta}{2} = \frac{r}{s-b},$$

$$\begin{aligned} \log r &= 11.09306 - 10 \\ \log(s-a) &= 1.34694 \quad (-) \\ \log\tan\frac{\alpha}{2} &= 9.74612 - 10, \end{aligned} \qquad \begin{aligned} \log r &= 11.09306 - 10 \\ \log(s-b) &= 1.14239 \quad (-) \\ \log\tan\frac{\beta}{2} &= 9.95067 - 10, \end{aligned}$$

$$\frac{\alpha}{2} = 29^\circ\,8', \qquad \frac{\beta}{2} = 41^\circ\,45',$$

$$\tan\frac{\gamma}{2} = \frac{r}{s-c}$$

$$\begin{aligned} \log r &= 11.09306 - 10 \\ \log (s-c) &= \underline{1.55328} \quad (-) \\ \log \tan\frac{\gamma}{2} &= 9.53978 - 10 \end{aligned}$$

$$\frac{\gamma}{2} = 19^\circ\, 7'.$$

Therefore

$$\begin{aligned} \alpha &= 58^\circ\, 16' \\ \beta &= 83^\circ\, 30' \\ \gamma &= \underline{38^\circ\, 14'} \\ \alpha + \beta + \gamma &= 180^\circ\, 0', \text{ which checks.} \end{aligned}$$

EXAMPLE 2. In the triangle ABC find β only, given

$$\begin{aligned} a &= 82.16 \\ b &= 70.67 \\ c &= \underline{48.25} \\ 2s &= 201.08, \quad s = 100.54. \end{aligned}$$

Adding,

$$\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)\,(s-c)}{ac}}.$$

$$s - a = 18.38 \qquad s - c = 52.29.$$

$$\begin{aligned} \log (s-a) &= 1.26435 & \log a &= 1.91466 \\ \log (s-c) &= \underline{1.71842} \quad (+) & \log c &= \underline{1.68350} \quad (+) \\ & 12.98277 - 10 & \log (ac) &= 3.59816 \\ \log (ac) &= \underline{3.59816} \quad (-) \\ & \div 2)\overline{19.38461 - 20} \end{aligned}$$

$$\log \sin\frac{\beta}{2} = 9.69230 - 10, \qquad \frac{\beta}{2} = 29^\circ\, 29.8', \qquad \beta = 59^\circ\, 0'.$$

EXERCISES

Solve the following triangles, and check.

1. $a = 6$, $b = 8$, $c = 4$. *Ans.* $\alpha = 46^\circ\, 40'$, $\beta = 104^\circ\, 20'$, $\gamma = 29^\circ\, 0'$.

2. $a = 11$, $b = 6$, $c = 9$.

3. $a = 14$, $b = 17$, $c = 9$. *Ans.* $\alpha = 55^\circ\, 20'$, $\beta = 92^\circ\, 40'$, $\gamma = 32^\circ\, 0'$.

4. $a = 120$, $b = 90$, $c = 150$.

5.	$a = 1694$, $b = 2043$, $c = 2907$.	*Ans.*	$\alpha = 34° 47'$, $\beta = 43° 30'$, $\gamma = 101° 44'$.	**6.**	$a = 37.62$, $b = 29.87$, $c = 49.51$.
7.	$a = 7439.5$, $b = 9062.4$, $c = 6174.3$.	*Ans.*	$\alpha = 54° 33.0'$, $\beta = 82° 54.6'$, $\gamma = 42° 32.4'$.	**8.**	$a = 576.49$, $b = 921.26$, $c = 502.98$.
9.	$a = 67.274$, $b = 38.665$, $c = 40.229$.	*Ans.*	$\alpha = 117° 0.2'$, $\beta = 30° 48.2'$, $\gamma = 32° 11.8'$.	**10.**	$a = 743.26$, $b = 597.49$, $c = 203.51$.

11. A body is acted on by a force of 723.4 pounds and by a second force of 529.5 pounds. The resultant force is 633.7 pounds. Find the angles that the second force and the resultant make with the first force. *Ans.* 121° 40′, 45° 20′.

12. Three forces of 269.8 pounds, 103.5 pounds, and 184.7 pounds act in such directions that they are in equilibrium. Draw a figure showing the directions of the forces, and find the angles between them to the nearest minute.

91. The area of a triangle. — Let S denote the area of the triangle ABC, and h its altitude from B.

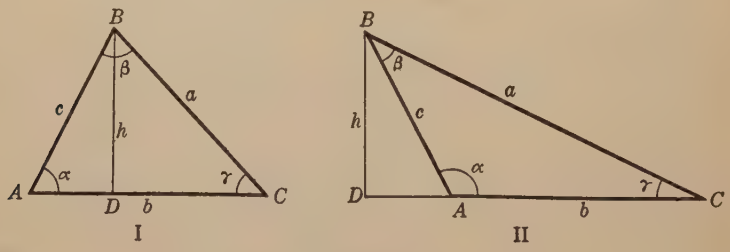

FIG. 70

Then $S = \frac{1}{2}bh$ and $h = c \sin \alpha$.

Therefore $S = \frac{1}{2}bc \sin \alpha$, or *the area of a triangle is equal to one-half the product of two sides times the sine of the included angle.*

$$S = \tfrac{1}{2}bc \sin \alpha, \qquad S = \tfrac{1}{2}ca \sin \beta, \qquad S = \tfrac{1}{2}ab \sin \gamma.$$

From this we can obtain another formula by using the law of sines:

$$c = \frac{b \sin \gamma}{\sin \beta}.$$

Then

$$S = \tfrac{1}{2}bc \sin \alpha = \frac{b^2 \sin \gamma \sin \alpha}{2 \sin \beta} = \frac{b^2 \sin \alpha \sin \gamma}{2 \sin (\alpha + \gamma)},$$

since $\sin \beta = \sin [180° - (\alpha + \gamma)] = \sin (\alpha + \gamma)$.

That is, *the area of a triangle is equal to one-half the square of any side times the product of the sines of the adjacent angles divided by the sine of the sum of those angles.*

$$S = \frac{a^2 \sin \beta \sin \gamma}{2 \sin (\beta + \gamma)}, \quad S = \frac{b^2 \sin \alpha \sin \gamma}{2 \sin (\alpha + \gamma)}, \quad S = \frac{c^2 \sin \alpha \sin \beta}{2 \sin (\alpha + \beta)}.$$

Still another formula can be found to express the area in terms of the sides. In the formula

$$S = \tfrac{1}{2}bc \sin \alpha,$$

set

$$\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}.$$

From Article 88,

$$\sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Therefore,

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore, also

$$S = sr,$$

where r has the meaning given to it in Article 89:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Now draw the inscribed circle of the triangle, and divide the triangle into three triangles by joining the vertices to the center of the circle. The area of one of the

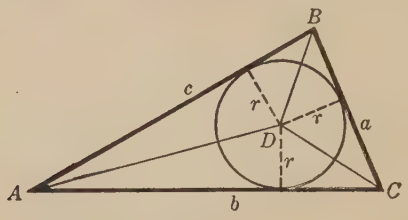

FIG. 71

triangles is equal to the radius times one-half the base, so that

$$S = s \times \text{(the radius of the inscribed circle)}.$$

This shows that r, as used in Article **89**, is equal to the radius of the inscribed circle.

EXAMPLE 1. Find the area of the triangle ABC, given

$$a = 9.015, \quad b = 10.234, \quad \gamma = 54^\circ\ 16'.$$

Set $S = \frac{1}{2}\, ab \sin \gamma$.

$$\begin{aligned} \log a &= 0.95497 \\ \log b &= 1.01005 \\ \log \sin \gamma &= \underline{9.90942 - 10\ (+)} \\ \log 2S &= 1.87444 \\ 2S &= 74.893 \\ S &= 37.446. \end{aligned}$$

EXAMPLE 2. Find the area of the triangle ABC, given

$$c = 14.32, \quad \alpha = 74^\circ\ 18', \quad \beta = 51^\circ\ 42'.$$

Set $S = \dfrac{c^2 \sin \alpha \sin \beta}{2 \sin (\alpha + \beta)}$. $\qquad \alpha + \beta = 126^\circ\ 0'.$

$$\begin{aligned} \log c &= 1.15594 \\ \log c &= 1.15594 \\ \log \sin \alpha &= 9.98349 - 10 \\ \log \sin \beta &= \underline{9.89475 - 10\ (+)} \\ & \quad 12.19012 - 10 \\ \log \sin (\alpha + \beta) &= \underline{9.90796 - 10\ (-)} \\ \log 2S &= 2.28216 \\ 2S &= 191.50 \\ S &= 95.75. \end{aligned}$$

EXAMPLE 3. Find the area of the triangle ABC, given

$$a = 68.44, \quad b = 79.83, \quad c = 52.79.$$

Set $S = \sqrt{s(s-a)(s-b)(s-c)}$.

$2s = 201.06$		
$s = 100.53$	$\log s =$	2.00230
$s - a = 32.09$	$\log (s-a) =$	1.50637
$s - b = 20.70$	$\log (s-b) =$	1.31597
$s - c = 47.74$	$\log (s-c) =$	1.67888 (+)
Add $s = 100.53$	$\div 2)$	6.50352
	$\log S =$	3.25176
	$S =$	1785.5.

EXERCISES

Find the areas of the following triangles. Work Examples **1** to **6** without logarithms.

1. $a = 7$, $b = 6$, $\gamma = 30°$. *Ans.* 10.5.

2. $a = 12$, $c = 8$, $\beta = 60°$.

3. $\alpha = 30°$, $\beta = 45°$, $c = 20$. *Ans.* 73.2.

4. $\beta = 40°$, $\gamma = 120°$, $a = 10$.

5. $a = 6$, $b = 9$, $c = 5$. *Ans.* 14.14.

6. $a = 13$, $b = 7$, $c = 8$.

7. $a = 869.4$, $b = 733.8$, $\gamma = 56° 27'$. *Ans.* 265,850.

8. $b = 21.48$, $c = 38.69$, $\alpha = 111° 24'$.

9. $b = 62.97$, $c = 41.32$, $\beta = 28° 19'$. *Ans.* 942.9.

10. $a = 468.3$, $c = 372.9$, $\alpha = 41° 49'$.

11. $a = 743.2$, $\alpha = 81° 18'$, $\beta = 62° 47'$. *Ans.* 145,750.

12. $c = 429.7$, $\alpha = 73° 11'$, $\beta = 55° 29'$.

13. $a = 629.7$, $b = 308.4$, $c = 411.2$. *Ans.* 54,092.

14. $a = 18.17$, $b = 3.24$, $c = 16.85$.

15. Find the area of the circle inscribed in the triangle the lengths of whose sides are 62.31, 20.43, 48.96. *Ans.* 129.56.

16. Find the area of the circle inscribed in the triangle the lengths of whose sides are 40.58, 31.69, 24.82.

EXERCISES AND APPLICATIONS

Solve the following triangles, find their areas, and check the solutions.

1. $a = 698.43$, $\beta = 113° 27.4'$, $\gamma = 24° 53.2'$. *Ans.* $\alpha = 41° 39.4'$, $b = 963.94$, $c = 442.19$, $S = 141660$.

2. $b = 24.839$, $\beta = 72° 44.7'$, $\gamma = 68° 17.2'$.

3. $\gamma = 28° 16.4'$, $b = 563.49$, $c = 384.17$.

Ans.

$\alpha = 107° 43.1'$, $\beta = 44° 0.5'$, $a = 772.58$, $S = 103100$;

$\alpha' = 15° 44.1'$, $\beta' = 135° 59.5'$, $a' = 219.94$, $S' = 29356$.

4. $a = 423.61$, $c = 593.72$, $\alpha = 41° 17.4'$.

5. $a = 302.54$, $c = 893.68$, $\beta = 124° 16.5'$. *Ans.* $\alpha = 13° 13.5'$, $\gamma = 42° 30.1'$, $b = 1092.8$, $S = 111710$.

6. $b = 11.49$, $c = 204.16$, $\alpha = 19° 24.7'$.

7. $a = 561.62$, $b = 238.49$, $c = 601.47$. *Ans.* $\alpha = 68° 54.6'$, $\beta = 23° 20.4'$, $\gamma = 87° 44.8'$, $S = 66917$.

8. $a = 748.87$, $b = 501.46$, $c = 295.21$.

9. A level road leads directly away from a cliff. From the top of the cliff the angle of depression of a milestone on the road is 6° 18′, and the angle of depression of the next nearer milestone is 57° 29′. Find the height of the cliff. *Ans.* 627 ft.

10. An enemy battery is on a hill above a plain. Its angle of elevation from a certain point in the plain is 8° 48′. From a point

823 feet nearer its angle of elevation is 13° 19′. How high is the battery above the plain?

11. An airplane is flying horizontally over a guiding line of boats which are anchored 2000 feet apart. As it approaches, its angle of elevation is observed simultaneously at two neighboring boats. At the nearer one it is 24°; at the other it is 18°. At what elevation is the plane flying (to the nearest 100 feet)? *Ans.* 2400 ft.

12. A farmhouse is separated by a grove from a road which runs straight east and west. It is visible, however, from two points on the road which are westward from the grove and 754 feet apart. From one of these points the house is 41° 20′ north of the road, and from the other it is 23° 40′ north of the road. How far is the farmhouse from the road?

13. A flagpole stands on a hill which is inclined at 18° to the horizontal. From a point 250 feet down the hill the angle of elevation of the top of the pole is 27°. How high is the flagpole? *Ans.* 44 ft.

14. A flagpole 36.6 feet high stands on a hill which has a slope of 16°. How long will its shadow be if, when the shadow extends directly down the hill, the altitude of the sun is 32°?

15. A lighthouse stands on a rock at a certain distance from a straight shore. It throws a beam of light which revolves uniformly. One-half second after illuminating the nearest point of the shore the beam reaches a point 400 feet along the shore, and one-half second later still, it reaches a point 600 feet still further along the shore. How far is the lighthouse from the shore, and how long does it take the light to make one revolution?

Ans. 895 ft., 7.5 secs.

16. A statue 12 feet high stands on a pedestal 9 feet high. How far from the pedestal is a point on the ground where the statue and pedestal subtend equal angles?

17. A line 620 feet long is drawn along one bank of a river. The angles which it makes with the lines of sight drawn to a point on the other bank are 47° 20′ and 63° 40′ respectively. How wide is the river? *Ans.* 438 ft.

18. A balloon floats directly above the line joining two points on

the ground. The two points are one mile apart. From one of them the angle of elevation of the balloon is 22° 10′, and from the other its angle of elevation is 29° 40′. Find the height of the balloon.

19. A tower 109 feet high stands on a cliff beside a river. From a point on the other side of the river and directly across from the tower the angle of elevation of the top of the tower is 38° 30′, and that of the bottom of the tower is 26° 30′. Find the width of the river.

Ans. 367 ft.

20. A northbound ship steams past a lighthouse. At a certain instant the bow of the vessel is 36° north of east and its stern is 27° north of east from the light. How far is the light from the path of the ship, if the ship's length is known to be 564 feet?

21. From a point 3.6 miles from one end of a lake, and 5.4 miles from the other end, the angle subtended by the lake is 104°. How long is the lake? *Ans.* 7.2 miles.

22. Two ends on a football team start from points on the line 25 yards apart and run down the field under a kick. They run directly toward the man who catches the ball and reach him just as he catches it, 5.5 seconds after they start. One can run 100 yards in 12 seconds, the other in 11 seconds. How far beyond the line of scrimmage did the ball go, and at what angle did the runners converge?

23. A and B are points on two islands. In order to find the distance AB from the shore, two points, C and D, are taken on shore 879 feet apart, and the angles at C and D are measured and found to be $\angle ACB = 79° 34'$, $\angle ACD = 24° 42'$, $\angle ADB = 82° 14'$, $\angle BDC = 38° 21'$. Find AB. *Ans.* 1464 ft.

24. Two points, A and B, which are 3100 feet apart, are directly below the line of flight of an airplane. At a certain instant when the plane is above the line AB its angle of elevation at A is 79° 20′, and its angle of elevation at B is 18° 50′. Just 15 seconds later, while it is still above AB, its angle of elevation at A is 21° 10′, and its angle of elevation at B is 49° 40′. What is its speed in miles per hour?

25. Surveyors establish five points, A, B, C, D, E. AB is measured and found to be 1698 feet. The bearings of the various lines are taken and found to be as follows.

AB: $S\ 26°\ 40'\ W$, CB: $N\ 73°\ 0'\ W$, DE: $S\ 21°\ 30'\ E$.
AC: $S\ 34°\ 30'\ E$, CD: $N\ 39°\ 50'\ E$,
AD: $N\ 83°\ 20'\ E$, CE: $N\ 82°\ 40'\ E$,

Find the length of DE. *Ans.* 2422 ft.

26. The course of a boat race is marked by five buoys, A, B, C, D, E. The first leg, AB, is 1010 feet long, and has the bearing $N\ 72°\ 40'\ W$. The bearings of the other legs are BC, $N\ 27°\ 50'\ E$; CD, $N\ 73°\ 40'\ E$; DE, $S\ 47°\ 20'\ E$; EA, $S\ 58°\ 10'\ W$. The bearing of C from A is $N\ 34°\ 10'\ W$, and that of D from A is $N\ 38°\ 10'\ E$. Find the length of the entire course.

27. Find the diameter of the largest circular circus tent that can be placed on a triangular lot whose sides are 73 feet, 64 feet and 85 feet in length. *Ans.* 41 ft.

28. A triangle is inscribed in a circle. Two of its sides are 123.9 cm. and 156.4 cm. in length, while the arc of which the third side is the chord is one-seventh of the circumference of the circle. Find the length of the third side and all of the angles.

29. A filling station is built on a triangular lot having a frontage of 175 feet on Main Street and 110 feet on First Avenue, which streets intersect obliquely. The rear line of the lot is 230 feet long. By how much will the area of the lot be increased if land is purchased so that the First Avenue frontage is increased to 200 feet, the Main Street frontage remaining unchanged? *Ans.* 7595 sq. ft.

30. A triangular lot has frontages of 78 feet and 94 feet and a rear line 69 feet long. If the 78 feet frontage is increased to 109 feet how much more will it cost to fence the new lot than it did to fence the old one, the cost of fencing being 45 cents a running foot?

31. A and B are points on opposite sides of a pond. The broken line $ACDB$ is run around the end of the pond, and it is found that $AC = 978$ feet, $CD = 1424$ feet, $DB = 2111$ feet, while $\angle CAD = 37°\ 44'$ and $\angle CBD = 29°\ 18'$. Find AB. *Ans.* 2672 ft.

32. We wish to find the distance between two points, A and B, which are hidden from each other. A straight line is run from B to a point C and continued to a point D from which A is visible.

By measurement $BC = 432$ yards, $CD = 587$ yards, $\angle BCA = 98° 20'$, $\angle BDA = 69° 40'$. Find AB.

33. When the elevation of the sun is such that a vertical rod 10 feet long casts a horizontal shadow 16 feet long, how long a shadow will be cast by a pole 24 feet long tipped 10° from the vertical away from the sun? *Ans.* 42 ft.

34. When a vertical rod 10 feet long casts a horizontal shadow 18 feet long, how long is a flagpole which, if tipped 12° from the vertical toward the sun, will cast a shadow 40 feet long?

35. A tower stands at a point A on an island. It is desired to find its height and its distance from a point B on the mainland. A point C is taken on the mainland 312 feet from B. The horizontal angles ABC and ACB are found to be $79° 42'$ and $70° 28'$, while the angle of elevation of the top of the tower from B is $20° 38'$. Find the height of the tower and the distance AB. *Ans.* 223 ft., 591 ft.

36. Work Ex. **35**, using $BC = 408$ feet, $ABC = 67° 23'$, $ACB = 59° 12'$, the angle of elevation $= 18° 16'$.

37. From the balcony of a fraternity house a flagpole 11 feet long leans forward so that its top is 9 feet farther forward than its foot. From a point on the ground directly in front of the pole, the angle of elevation of the top of the pole is $28° 11'$ and that of the foot of the pole is $19° 24'$. Find the height of the balcony and its horizontal distance from the point of observation. *Ans.* 21.4 ft., 60.7 ft.

38. At the top of a hill which slopes $17° 34'$ from the horizontal, stands a pole 49.3 feet long, which is tipped $9° 15'$ from the vertical toward the bottom of the hill. From the bottom of the hill the angle of elevation of the top of the pole is $21° 29'$. Find the height of the hill.

39. A tower of a power transmission line is 102 feet high. It stands on top of a hill. From a point 634 feet down the hill from the tower the angle subtended by the tower is $8° 47'$. How steep is the hill? *Ans.* $9° 34'$.

40. The Eiffel Tower is 300 meters high. The top of the Arch of Triumph is 54 meters higher than the base of the tower. From the top of the arch the tower subtends an angle of $10° 10'$. Find the horizontal distance from the arch to the tower.

41. An airplane is flying in a straight, horizontal line at the rate of 120 miles per hour. A person directly below the path of the plane observes it just after it has passed overhead. Its angle of elevation is 82° 30′. One minute later its angle of elevation is 20° 20′. At what height is it flying? *Ans.* 4114 ft.

42. Work Ex. **41**, using the two angles of elevation 74° 20′ and 34° 15′, and assuming that between the times of the two observations the plane passes directly over the observer.

43. A battleship is anchored off a straight shore which runs due north and south. There are two observing stations on the shore 3 miles apart. From one the bearing of the battleship is N 38° 20′ E; and its bearing from the other is N 61° 30′ E. How far is the ship from the shore? *Ans.* 4.16 miles.

44. Work Ex. **43**, making the bearings N 78° 40′ E and S 74° 10′ E.

45. A battleship which is steaming steadily along a straight course is observed simultaneously from two stations, one of which is 3 miles due north of the other. At one instant its bearings from the two stations are N 36° 40′ E and N 17° 30′ E. Twelve minutes later its bearings from the stations are S 68° 10′ E and N 57° 40′ E. Find the speed of the battleship. *Ans.* 17.9 miles per hour.

46. Work Ex. **45**, using the bearings N 42° 40′ E and N 21° 20′ E at the first, and S 58° 30′ E and N 78° 10′ E at the second observation.

47. A torpedo boat enforcing a blockade is 11 miles N 62° W from an enemy port. A vessel sails from the port in the direction S 22° W at the rate of 14 miles per hour. At what rate and in what direction should the torpedo boat steam in order to overtake the vessel 8 miles from port? *Ans.* S 23° 57′ E, 22.6 miles per hour.

48. Two boys wish to find the distance from a point A on the shore to a point B on an island, but have no instruments for measuring angles. They choose a point C on shore 438 feet from A, and prolong BA to D and BC to E. By measurement $AD = 498$ feet, $AE = 833$ feet, $CE = 524$ feet, $CD = 679$ feet. Find AB.

49. In any triangle, prove that

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}\gamma}.$$

Suggestion: Extend AC to D, making AD equal to $(a + b)$. Apply the sine

law to triangle ABD, and note that

$$\beta + \tfrac{1}{2}\gamma = \beta + \tfrac{1}{2}(180° - \alpha - \beta) = 90° + \tfrac{1}{2}(\beta - \alpha).$$

Note: This formula may be used conveniently to check any solution, since it involves all of the parts and is suitable for logarithmic work.

50. In any triangle, prove that

$$\frac{a - b}{c} = \frac{\sin \tfrac{1}{2}(\alpha - \beta)}{\cos \tfrac{1}{2}\gamma}.$$

51. By the formulas of Ex. **49** and Ex. **50** prove the law of tangents.

52. Use the formulas of Ex. **49** and Ex. **50** to check the solutions of the triangles of Ex. **1** to Ex. **8**.

53. A ladder leans against a building and makes an angle of 42° 30′ with the horizontal. When its foot is moved 16 feet nearer the building it makes an angle of 72° 10′ with the horizontal. How much higher does it reach in the second position than in the first?

Ans. 10 ft., 3 ins.

54. A ladder leans against a house, and makes an angle of 37° 12′ with the horizontal. When its foot is moved 10.4 feet nearer the house it makes an angle of 78° 33′ with the horizontal. Find the length of the ladder.

55. In the roof of an athletic field house three steel girders, 27.4 feet, 39.6 feet, and 50.5 feet, in length, are to be riveted into a triangle. The rivet holes must be made so as to allow the girders to meet at what angles?

Ans. 32° 39′, 51° 14′, 96° 7′.

56. The sides of a triangle are 16.53 meters, 11.49 meters, and 21.37 meters in length. Find the area of the inscribed circle.

57. The sides of a triangle are 9.58 meters, 12.21 meters, and 15.35 meters. Find the radius of the circumscribed circle.

Ans. 7.68 meters.

58. The sides of a triangle are 21.47, 32.68, 47.11. Find the area of the circumscribed circle.

59. Show that the radius of the circle circumscribed about a triangle whose sides are a, b, c, is

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

60. Show that the product of the sides of a triangle is equal to twice the perimeter of the triangle multiplied by the radii of the inscribed and circumscribed circles.

61. Surveyors are running a straight line AB, which is stopped by an obstacle at B. In order to prolong the line beyond the obstacle, at B they turn 49° 17′ to the right and measure 334.6 feet to a point C. They then turn 78° 43′ to the left, and measure a certain distance until they reach the prolongation of AB at a point D. What must this distance CD be, and what is the length of the gap BD?

Ans. 516.1 ft., 667.7 ft.

62. Work Ex. **61**, making $BC = 521.7$ feet, $ABC = 124° 12'$, and $BCD = 63° 18'$.

63. Without instruments for measuring angles, it is desired to find the distance between two points, A and B, which are separated by an obstacle. A third point, C, is taken 269 feet from A and 362 feet from B. A point D is taken on AC 104 feet from C, and E is taken on BC 93 feet from C. By measurement $DE = 121$ feet. Find AB.

Ans. $393\frac{1}{2}$ ft.

64. Work Ex. **63**, taking C 184 feet from A and 274 feet from B. Let $DC = 67$ feet, $EC = 58$ feet, and $DE = 84$ feet.

65. In mapping a lake, it is desired to find the distance between two points A and B on two islands. There is only one point, C, on the mainland from which both A and B are visible. A point D from which A is visible is chosen, and the line DC is continued past C to a point E from which B is visible. By measurement, $DCA = 48° 10'$, $CDA = 74° 40'$, $BCE = 58° 10'$, $BEC = 80° 20'$, $DC = 643$ feet, $CE = 897$ feet. Find AB and the angle that AB makes with the direction DC.

Ans. 1394 ft., 43° 30′.

66. Work Ex. **65**, using the following measurements. $DCA = 28° 50'$, $CDA = 98° 0'$, $BCE = 112° 20'$, $BEC = 54° 40'$, $DC = 746$ feet, $CE = 564$ feet.

CHAPTER X

THE INVERSE TRIGONOMETRIC FUNCTIONS

92. Inverse functions. — The equation

$$y = \log_a x$$

enables us to find the value of y when the value of x is known; that is, it *expresses y as a function of x*. But it also enables us to find x when the value of y is known, since it is equivalent to the equation

$$x = a^y.$$

It therefore *expresses x as a function of y*. When two variables, x and y, are connected by some law which enables us to express y as one function of x and x as some (in general different) function of y, those two functions are said to be *inverse functions*, each of the other. Thus, the logarithmic function is the inverse of the exponential function, since the equations

$$y = \log_a x \quad \text{and} \quad x = a^y$$

express the same relation between x and y.

93. The inverse trigonometric functions. — If x and y are related by the equation

$$y = \sin x,$$

y is a certain function (namely the sine function) of x, and x is a certain function (the inverse of the sine function) of y.

Definition: If y is the sine of x, we say that x is the *inverse sine* of y. There are two common notations for this:

$$x = \arcsin y \quad \text{and} \quad x = \sin^{-1} y.$$

The student should be familiar with both of these notations, for he may meet with either. Here we shall use the form $x = \arcsin y$.

Evidently if $y = \sin x$ (or if $x = \arcsin y$, which means the same thing), x is *an angle whose sine is* y. The equation $x = \arcsin y$ may be read in any of the following ways: x is equal to the *inverse sine* of y, or to the *arc-sine* of y, or to the *anti-sine* of y. But better than any of these, for the present, is the phrase: x is equal to *an angle whose sine is* y, for this reading carries its own definition.

The student should frame analogous definitions for $\arccos y$, $\arctan y$, $\operatorname{arccot} y$, $\operatorname{arcsec} y$, and $\operatorname{arccsc} y$.

94. Values for which the inverse trigonometric functions are defined. — Since the sine function takes on only the values from -1 to 1, the inverse sine function, $\arcsin x$, is defined only for values of x from $x = -1$ to $x = 1$. For example, $\arcsin 2$ is not defined since there is no angle whose sine is 2. On the other hand, $\arctan x$ is defined for all values of x, since the tangent assumes all values.

EXERCISE. Write the values of x for which each of the trigonometric functions of x is defined.

EXAMPLE 1. Find $\arcsin \frac{1}{2}$. Here we wish to find "an angle whose sine is $\frac{1}{2}$." Since $\sin \frac{\pi}{6} = \frac{1}{2}$, $\arcsin \frac{1}{2} = \frac{\pi}{6}$.

EXAMPLE 2. Find $\arctan (-1)$. Since $\tan \frac{3\pi}{4} = -1$, $\arctan (-1) = \frac{3\pi}{4}$.

EXAMPLE 3. Find $\arcsin (\frac{1}{2}\sqrt{3}) + \arctan (1/\sqrt{3})$. An angle whose sine is $\frac{1}{2}\sqrt{3}$ is $\pi/3$; an angle whose tangent is $(1/\sqrt{3})$ is $\pi/6$. Therefore

$$\arcsin (\tfrac{1}{2}\sqrt{3}) + \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}.$$

EXAMPLE 4. Show that $\arctan x = \operatorname{arccot} (1/x)$. If we let

$$y = \arctan x,$$

then, by definition,

$$x = \tan y.$$

Therefore

$$\cot y = 1/x.$$

Consequently

$$\operatorname{arccot} (1/x) = y = \arctan x.$$

EXAMPLE 5. Show that $\arcsin x = \arccos \sqrt{1 - x^2}$.

Set

$$\theta = \arcsin x,$$

so that

$$x = \sin \theta.$$

Then

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2},$$

and

$$\arccos \sqrt{1 - x^2} = \theta = \arcsin x.$$

EXAMPLE 6. Show that $\arcsin x + \arccos x = \pi/2$.

Note: As will be seen in Article **95** this sum has many different values. We are to prove here that, no matter what allowable value is assigned to x there is *one* value of the sum which is equal to $\pi/2$. A similar remark applies to Example 3, above.

Whatever the value of θ, we know that if we set

$$\phi = \frac{\pi}{2} - \theta,$$

then $\cos \phi = \sin \theta$. If we now take $\theta = \arcsin x$, we shall have

$$x = \sin \theta = \cos \phi,$$

so that $\phi = \arccos x$. But since $\theta + \phi = \pi/2$,

$$\arcsin x + \arccos x = \pi/2.$$

Suggestion: In a problem involving inverse trigonometric functions, if no other solution occurs to you, substitute some letter, just as θ, ϕ, and y were substituted in these examples, for each of the inverse functions involved. The problem may then usually be restated in terms of the trigonometric functions instead of their inverses.

EXERCISES

Find the values of the following numbers.

1. $\arcsin 1$.	*Ans.* $\pi/2$.	**2.** $\arccos (-1)$.
3. $\operatorname{arcsec} 2$.	*Ans.* $\pi/3$.	**4.** $\operatorname{arccot} \sqrt{3}$.
5. $\arcsin (-1)$.	*Ans.* $-\pi/2$.	**6.** $\operatorname{arccsc} \frac{2}{\sqrt{3}}$.
7. $\arctan 0$.	*Ans.* 0.	**8.** $\operatorname{arccot} 0$.
9. $\arccos \frac{\sqrt{3}}{2}$.	*Ans.* $\pi/6$.	**10.** $\arctan 1$.

11. arcsin (sin y). *Ans.* y. **12.** arctan (tan x).
13. sin (arcsin x). *Ans.* x. **14.** cos (arccos y).
15. cos (arcsin x). *Ans.* $\sqrt{1 - x^2}$. **16.** sin (arccos z).
17. tan (arcsec x). *Ans.* $\sqrt{x^2 - 1}$. **18.** sec (arctan x).
19. cot (arctan x). *Ans.* $1/x$. **20.** csc (arcsin x).

95. Multiple values of the functions. — Since there are many angles whose sine is equal to $\frac{1}{2}$, it is evident that arcsin $\frac{1}{2}$ is not defined uniquely. It may have any one of the values $\pi/6$, $5\pi/6$, $-7\pi/6$, $-11\pi/6$, $13\pi/6$, etc. And, in general, if α is a value of arcsin x, so is $(\pi - \alpha)$, and so is any angle obtained from α or $(\pi - \alpha)$ by adding or subtracting any multiple of 2π. We say that the inverse trigonometric functions are *multiple-valued functions*, since, corresponding to any value of the variable, for which they are defined at all, there are many values of each function.

96. Principal values. — To avoid confusion, out of the infinitely many values of an inverse trigonometric function we select one, which is called the *principal value* of the function. For a given x, the principal value of an inverse trigonometric function of x is the value which is numerically the smallest; but if a positive and a negative value are numerically equal but smaller than any other, then the positive value is the principal one.

The principal values of arcsin x, arctan x, arccot x and arccsc x range in value from $-\pi/2$ to $\pi/2$, while the principal value of arccos x and arcsec x ranges from 0 to π.

Thus the principal value of

$$\arcsin \tfrac{1}{2} = \frac{\pi}{6},$$

for $\pi/6$ is numerically the smallest value of the function. And the principal value of

$$\arccos \tfrac{1}{2} = \frac{\pi}{3},$$

for, though $\pi/3$ and $-\pi/3$ are two values which are numerically equal, preference is to be given to the positive value.

When we write an inverse trigonometric function *we shall*, from now on, *suppose that it stands for its principal value*, unless there is reason to suppose otherwise.

EXERCISES

Write the principal value of each of the following quantities.

1. $\arcsin(-\frac{1}{2})$. *Ans.* $-\pi/6$. **2.** $\arccos(-\frac{1}{2}\sqrt{3})$.
3. $\arctan(-1)$. *Ans.* $-\pi/4$. **4.** $\operatorname{arccot}(-1)$.
5. $\arccos 0$. *Ans.* $\pi/2$. **6.** $\arcsin 1$.
7. $\operatorname{arcsec}(-1)$. *Ans.* π. **8.** $\operatorname{arcsec}(-2)$.
9. $\operatorname{arccot} 1$. *Ans.* $\pi/4$. **10.** $\arctan 0$.
11. $\operatorname{arccsc}(-2)$. *Ans.* $-\pi/6$. **12.** $\operatorname{arccsc}(-1)$.

Write the value of each of the following expressions.

13. $\sin \arcsin(-1)$. *Ans.* -1. **14.** $\cos \arccos \frac{1}{2}$.
15. $\cos \arcsin(-\frac{1}{2})$. *Ans.* $\frac{1}{2}\sqrt{3}$. **16.** $\tan \arcsin(-\frac{1}{2})$.
17. $\tan \arcsin(-\frac{1}{2}\sqrt{3})$. *Ans.* $-\sqrt{3}$. **18.** $\cot \arccos(-\frac{1}{2})$.
19. $\tan \arccos(-\frac{1}{2})$. *Ans.* $-\sqrt{3}$. **20.** $\sec \arcsin(-\frac{1}{2})$.
21. $\sec \arcsin(-\frac{1}{2}\sqrt{2})$. *Ans.* $\sqrt{2}$. **22.** $\csc \arctan 1$.
23. $\sin \arctan(-1)$. *Ans.* $-\frac{1}{2}\sqrt{2}$. **24.** $\sin \arccos(-\frac{1}{2}\sqrt{3})$.
25. $\arcsin(\sin 45°)$. *Ans.* $45°$. **26.** $\arccos(\cos 112°)$.
27. $\arcsin(\sin 120°)$. *Ans.* $60°$. **28.** $\arccos(\cos 210°)$.
29. $\arctan(\sin 90°)$. *Ans.* $45°$. **30.** $\operatorname{arccot}(\sin 270°)$.

31. $\sin(\arcsin x + \arccos x)$. *Ans.* 1.
32. $\sin(\arccos x - \arcsin x)$.
33. $\sin(\arcsin x + \arcsin y)$. *Ans.* $x\sqrt{1-y^2} + y\sqrt{1-x^2}$.
34. $\cos(\arcsin x + \arcsin y)$.
35. $\sin(\arcsin x + \arccos y)$. *Ans.* $xy + \sqrt{(1-x^2)(1-y^2)}$.
36. $\cos(\arcsin x - \arccos y)$.
37. $\sin(2 \arcsin x)$. *Ans.* $2x\sqrt{1-x^2}$.
38. $\cos(2 \arcsin x)$.
39. $\sin(3 \arcsin x)$. *Ans.* $3x - 4x^3$.
40. $\cos(3 \arccos x)$.
41. $\sin(4 \arcsin x)$. *Ans.* $\sqrt{1-x^2}\,(4x - 8x^3)$.
42. $\cos(4 \arccos x)$.

43. $\tan (2 \arctan x)$. *Ans.* $2x/(1 - x^2)$.

44. $\tan (\arctan x - \arctan y)$.

45. $\arctan\left(\dfrac{2 \tan x}{1 - \tan^2 x}\right)$. *Ans.* $2x$.

46. $\arcsin (2 \sin x \cos x)$.

47. $\arccos \sqrt{\dfrac{1 + \cos x}{2}}$. *Ans.* $x/2$.

48. $\arccos (2 \cos^2 x - 1)$.

★**97. Graphs of the inverse trigonometric functions.** — Since the equations

$$x = \sin y \quad \text{and} \quad y = \arcsin x$$

express exactly the same relation between x and y, the graph of the function

$$y = \arcsin x$$

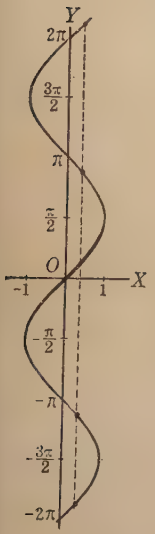

Fig. 72

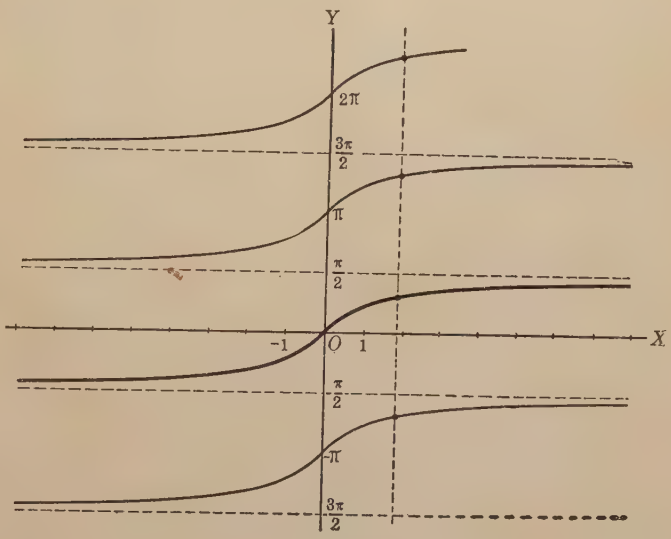

Fig. 73

is precisely the same as the graph of

$$y = \sin x$$

except that the x and y axes are interchanged with respect to the curve. The graph is shown in Figure 72. The intersections of the curve with any vertical line give the values of the arcsine for the corresponding value of x. The principal values lie on the heavy line. The graph of the function

$$y = \arctan x$$

is shown in Figure 73. The principal values lie on the heavy line.

EXERCISE: Sketch the graphs of the other inverse trigonometric functions, indicating the principal values with a heavy line.

EXAMPLE 1. Prove the identity

$$3 \arcsin x = \arcsin (3x - 4x^3).$$

Set $\theta = \arcsin x$, so that $\sin \theta = x$. Then, since

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta,$$

for any angle θ,

$$\begin{aligned} 3 \arcsin x = 3\theta &= \arcsin (3 \sin \theta - 4 \sin^3 \theta) \\ &= \arcsin (3x - 4x^3). \end{aligned}$$

EXAMPLE 2. Solve for x the equation

$$\arcsin x + \arcsin 2x = 90°.$$

Let $\theta = \arcsin x$ and $\phi = \arcsin 2x$, and take the sine of each side of the equation. We get

$$\sin (\theta + \phi) = 1 \quad \text{or} \quad \sin \theta \cos \phi + \cos \theta \sin \phi = 1.$$

But $\sin \theta = x$, $\cos \theta = \sqrt{1 - x^2}$, $\sin \phi = 2x$, $\cos \phi = \sqrt{1 - 4x^2}$. Therefore

$$x\sqrt{1 - 4x^2} + 2x\sqrt{1 - x^2} = 1. \tag{1}$$

After squaring and collecting terms we reduce this to the equation

$$5x^2 - 1 = 0. \tag{2}$$

Therefore

$$x = \pm \tfrac{1}{5} \sqrt{5} = \pm .4472.$$

However $-\frac{1}{5}\sqrt{5}$ does not satisfy equation (1), so we retain only the value $x = .4472$.

To check this result we find from a table of sines that

$$\arcsin x = \arcsin (.4472) = 26° 34',$$
$$\arcsin 2x = \arcsin (.8944) = 63° 26'.$$

Therefore, if $x = .4472$,

$$\arcsin x + \arcsin 2x = 90°.$$

EXERCISES

Prove the following identities.

1. $2 \arcsin x = \arccos (1 - 2x^2)$.
2. $2 \operatorname{arccot} x = \arctan \dfrac{2x}{x^2 - 1}$.
3. $2 \arccos x = \operatorname{arcsec} \dfrac{1}{2x^2 - 1}$.
4. $2 \arccos x = \operatorname{arccsc} \dfrac{1}{2x\sqrt{1 - x^2}}$.
5. $\arctan x + \arctan y = \arctan \dfrac{x + y}{1 - xy}$.
6. $\arcsin x + \arcsin y + \arcsin z$
$= \arcsin [x\sqrt{(1 - y^2)(1 - z^2)} + y\sqrt{(1 - z^2)(1 - x^2)}$
$+ z\sqrt{(1 - x^2)(1 - y^2)} - xyz]$.
7. $\arccos x + \arccos y + \arccos z$
$= \arccos [xyz - x\sqrt{(1 - y^2)(1 - z^2)} - y\sqrt{(1 - z^2)(1 - x^2)}$
$- z\sqrt{(1 - x^2)(1 - y^2)}]$.
8. $\arcsin x + \arcsin y + \arcsin z$
$= \arccos [\sqrt{(1 - x^2)(1 - y^2)(1 - z^2)} - xy\sqrt{1 - z^2}$
$- yz\sqrt{1 - x^2} - zx\sqrt{1 - y^2}]$.

Solve the following equations for x.

9. $\arctan 2x = \pi/4 - \arctan x$. *Ans.* .2808.
10. $\arcsin 2x = \pi/4 - \arcsin x$.
11. $\arccos (2x^2 - 1) = 2 \arccos \frac{1}{2}$. *Ans.* .5.
12. $\arccos (1 - 2x^2) = 2 \arcsin \frac{3}{4}$.
13. $\arcsin x = \pi/2 - \arctan x$. *Ans.* .786.
14. $\arccos x = \pi/2 - \arctan x$.

★CHAPTER XI

DE MOIVRE'S THEOREM AND SERIES EXPANSIONS

★98. Polar coördinates. — The position of a point in a plane is determined if its rectangular coördinates (x, y) are known. An alternative method of stating the position of a point is to give its *polar coördinates*. These are r, which is the length of the line drawn from the origin or *pole* to the point, and ϕ, which is the angle that this line makes with the x-axis or *polar axis*. The coördinates are written (r, ϕ); r is called the *radius vector* of the point; ϕ is called its *vectorial angle*. The radius vector is expressed in any convenient unit of length, and the vectorial angle is commonly expressed in radians, though degrees may be used. The distance r, if positive, is measured from the pole along the terminal line of the angle ϕ. If r is negative, it is to be measured in the opposite direction, that is, from the origin or pole but along the line made by extending the terminal side of ϕ through the pole.

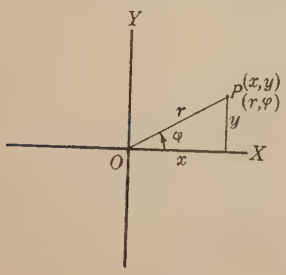

Fig. 74

In the figure, the points $(6, 2\pi/3)$ and $(-6, 2\pi/3)$ are shown.

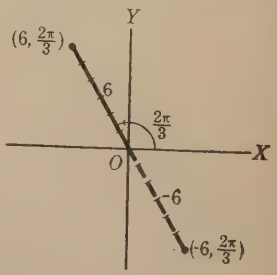

Fig. 75

If (x, y) and (r, ϕ) are respectively the rectangular and the polar coördinates of the same point, we clearly have the following relations, by which to convert from one system to the other,

$$x = r\cos\phi, \qquad r = \sqrt{x^2 + y^2},$$

$$y = r\sin\phi, \qquad \phi = \arctan\frac{y}{x}.$$

EXAMPLE 1. Find polar coördinates of the point whose rectangular coördinates are $(1, \sqrt{3})$. For this point

$$r = \sqrt{1 + 3} = 2,$$

$$\phi = \arctan\frac{\sqrt{3}}{1} = \frac{\pi}{3}.$$

The required coördinates are therefore $(2, \pi/3)$.

EXAMPLE 2. Find the rectangular coördinates of the point whose polar coördinates are $(5, 4\pi/3)$. We have

$$x = r\cos\phi = 5\cos\frac{4\pi}{3} = 5\left(-\tfrac{1}{2}\right) = -\tfrac{5}{2},$$

$$y = r\sin\phi = 5\sin\frac{4\pi}{3} = 5\left(-\tfrac{1}{2}\sqrt{3}\right) = -\tfrac{5}{2}\sqrt{3}.$$

The required coördinates are $(-\frac{5}{2}, -\frac{5}{2}\sqrt{3})$.

It should be particularly noted that, while each point has just one set of rectangular coördinates, there are infinitely many sets of polar coördinates for each point. For there are infinitely many values of $\arctan(y/x)$, and consequently infinitely many values of ϕ, differing by multiples of 2π. Thus, in Example 1, above, the coördinates may be taken as $(2, \pi/6 \pm 2n\pi)$ where n is any integer.

To plot the point (r, ϕ), first lay off the angle ϕ from the polar axis, and then, along the terminal side of ϕ, measure off the distance r from the pole (or the distance $-r$, along the terminal side extended through the pole, if r is negative).

EXERCISES

Plot the points having the following polar coördinates and find their rectangular coördinates.

1. $(3, \pi/2)$. *Ans.* $(0, 3)$. **2.** $(4, \pi)$.

3. $(4, 0)$. *Ans.* $(4, 0)$. **4.** $(2, -4\pi)$.

5. $(5, 7\pi/6)$. *Ans.* $(-5\sqrt{3}/2, -5/2)$. 6. $(4, 5\pi/3)$.
7. $(1, -9\pi/2)$. *Ans.* $(0, -1)$. 8. $(0, 6\pi)$.
9. $(0, 4)$. *Ans.* $(0, 0)$. 10. $(5, 0)$.
11. $(3, 1/2)$. *Ans.* $(2.63, 1.44)$. 12. $(4, -2/3)$.
13. $(-4, \pi/2)$. *Ans.* $(0, -4)$. 14. $(-2, \pi)$.
15. $(-2, \pi/6)$. *Ans.* $(-\sqrt{3}, -1)$. 16. $(-3, 3\pi/4)$.
17. $(-4, 4\pi/3)$. *Ans.* $(2, 2\sqrt{3})$. 18. $(-6, 11\pi/6)$.

For the following points the rectangular coördinates are given. Find one set of polar coördinates for each point, and plot the point.

19. $(0, 6)$. *Ans.* $(6, \pi/2)$. 20. $(7, 0)$.
21. $(-3, 0)$. *Ans.* $(3, \pi)$. 22. $(0, -4)$.
23. $(2, 3)$. *Ans.* $(\sqrt{13}, \arctan \frac{3}{2})$. 24. $(4, 1)$.
25. $(-12, 5)$. *Ans.* $(13, \arctan \frac{5}{12})$. 26. $(3, -4)$.
27. $(0, 0)$. *Ans.* $(0, \phi)$. 28. $(\pi, \pi/2)$.

★**99. Complex numbers.** — We shall use the letter i to denote the imaginary number, $i = \sqrt{-1}$. We then have

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1, \text{ etc.}$$

If a and b are two real numbers, $a + bi$ is called a *complex number*, of which a is the *real part* and bi is the *imaginary part*. Complex numbers are treated algebraically according to the following rules: Let $a + bi$ and $c + di$ be two complex numbers. Then

I. If $a + bi = c + di$, $a = c$ and $b = d$; or, *if two complex numbers are equal, their real parts are equal and their imaginary parts are equal.*

II. $(a + bi) + (c + di) = (a + c) + i(b + d)$.

III. $(a + bi)(c + di) = (ac - bd) + i(bc + ad)$.

IV. $$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}.$$

These rules are best applied by operating with i as though it were a real number and, in the end, replacing i^2 by -1.

***100. Graphical representation of complex numbers.** — The complex number $x + iy$ may be represented graphically by the point (x, y). Thus, the point whose rectangular coördinates are $(3, -7)$ represents the complex number $3 - 7i$. Suppose that (r, ϕ) are the polar coördinates of the point (x, y). Then

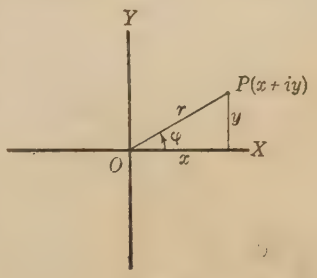

FIG. 76

$$r = \sqrt{x^2 + y^2}$$

is said to be the *modulus* or *absolute value* of the complex number $(x + iy)$. This may also be written

$$r = |x + iy|.$$

And

$$\phi = \arctan \frac{y}{x}$$

is called the *amplitude* or *angle* of the number $(x + iy)$.

Since $x = r \cos \phi$ and $y = r \sin \phi$, we can write

$$x + iy = r(\cos \phi + i \sin \phi).$$

Still another way of representing the complex number $(x + iy)$ is to picture it as the *vector* OP (Article **43**). It is not essential, in this case, that the vector extend from the point O. Provided that it has the length r and makes with OX an angle ϕ, a vector represents the number $r(\cos \phi + i \sin \phi)$, no matter what its initial point.

EXERCISES

Find the modulus and amplitude of each of the following complex numbers. Represent each number graphically as a point and also as a vector.

1. $3 + 4i$. *Ans.* $r = 5, \phi = \arctan \frac{4}{3}$.

2. $4 - 3i$.

3. i. *Ans.* $r = 1, \phi = \pi/2 \pm 2n\pi$.

4. $-4i$.

5. $5 - 3i$. *Ans.* $r = \sqrt{34}, \phi = \arctan(-\frac{3}{5})$.

6. $-7 + 8i$.

7. -2. *Ans.* $r = 2, \phi = \pi \pm 2n\pi$.

8. $1 + i$.

Given the following pairs of values of r and ϕ, write the corresponding complex numbers. Draw the figure.

9. $r = 4, \phi = \pi/6$. *Ans.* $2\sqrt{3} + 2i$. **10.** $r = 8, \phi = 3\pi/4$.
11. $r = 7, \phi = 3\pi/2$. *Ans.* $-7i$. **12.** $r = 4, \phi = -\pi$.
13. $r = 13, \phi = \arctan(5/12)$. *Ans.* $12 + 5i$.
14. $r = 34, \phi = \arctan(-8/15)$.
15. $r = 4, \phi = 2$. *Ans.* $4(\cos 2 + i \sin 2)$. **16.** $r = 6, \phi = 0$.

★**101. Multiplication of complex numbers.** — Let z_1 and z_2 be any two complex numbers, say,

$$z_1 = r_1(\cos \phi_1 + i \sin \phi_1),$$
$$z_2 = r_2(\cos \phi_2 + i \sin \phi_2).$$

By Article **99**, Rule III, their product is

$$\begin{aligned} z_1 z_2 &= r_1(\cos \phi_1 + i \sin \phi_1) \cdot r_2(\cos \phi_2 + i \sin \phi_2) \\ &= r_1 r_2[(\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2) + i(\sin \phi_1 \cos \phi_2 + \cos \phi_1 \sin \phi_2)] \\ &= r_1 r_2[\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)]. \end{aligned}$$

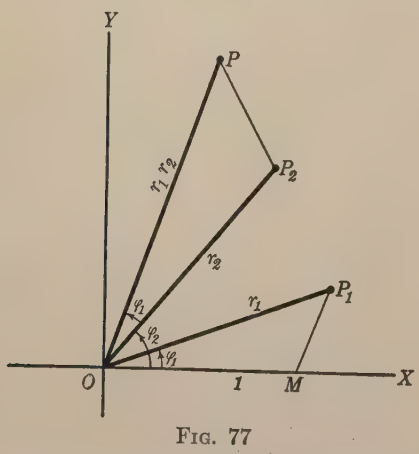

FIG. 77

This is a complex number whose modulus is $r_1 r_2$ and whose amplitude is $(\phi_1 + \phi_2)$. Therefore *the product of two complex numbers is a number whose modulus is the product of the moduli of the two factors and whose amplitude is the sum of the amplitudes of the two factors.*

The product may be constructed in the following way. Let P_1 and P_2 be the points representing z_1 and z_2. Let M be the point 1. Then construct the triangle OP_2P similar to OMP_1. The point P then represents the product $z_1 z_2$. For the angle P_2OP is equal to angle MOP_1

and therefore to ϕ_1, so that angle MOP is equal to $(\phi_1 + \phi_2)$. And, from the similarity of the triangles,

$$\frac{\overline{OP}}{\overline{OP_1}} = \frac{\overline{OP_2}}{\overline{OM}}.$$

Therefore, since $OM = 1$,

$$\overline{OP} = \overline{OP_1} \cdot \overline{OP_2} = r_1 \cdot r_2.$$

Applying the law of multiplication to n factors, we find:

$$[r_1(\cos\phi_1 + i\sin\phi_1)]\,[r_2(\cos\phi_2 + i\sin\phi_2)] \cdot \cdots \cdot [r_n(\cos\phi_n + i\sin\phi_n)]$$
$$= r_1 r_2 \cdot \cdots \cdot r_n[\cos(\phi_1 + \phi_2 + \cdots + \phi_n) + i\sin(\phi_1 + \phi_2 + \cdots + \phi_n)].$$

★**102. De Moivre's Theorem.** — If we set each of the n factors of the preceding formula equal to $r(\cos\phi + i\sin\phi)$, it becomes

$$[r(\cos\phi + i\sin\phi)]^n = r^n(\cos n\phi + i\sin n\phi).$$

In particular, if $r = 1$,

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi.$$

This is known as *De Moivre's Theorem.*

We have proved De Moivre's Theorem in case n is any positive integer. We shall now prove that it is true if n is any rational number, positive or negative, integral or fractional.

In case n is a negative integer, set $n = -m$ where m is a positive integer. Then

$$(\cos\phi + i\sin\phi)^n = (\cos\phi + i\sin\phi)^{-m}$$
$$= \frac{1}{(\cos\phi + i\sin\phi)^m} = \frac{1}{\cos m\phi + i\sin m\phi}$$
$$= \frac{\cos m\phi - i\sin m\phi}{\cos^2 m\phi + \sin^2 m\phi} = \cos m\phi - i\sin m\phi$$
$$= \cos(-m\phi) + i\sin(-m\phi) = \cos n\phi + i\sin n\phi.$$

If n is a rational fraction, say $n = p/q$,

$$(\cos\phi + i\sin\phi)^n = (\cos\phi + i\sin\phi)^{\frac{p}{q}}.$$

This is equal to some complex number. Set

$$(\cos\phi + i\sin\phi)^{\frac{p}{q}} = \rho(\cos\theta + i\sin\theta).$$

Take the qth power of each side. Since q is an integer,

$$(\cos \phi + i \sin \phi)^p = \rho^q (\cos q\theta + i \sin q\theta)$$

and

$$\cos p\phi + i \sin p\phi = \rho^q (\cos q\theta + i \sin q\theta).$$

Equating real and imaginary parts, we see that $\rho = 1$, and

$$q\theta = p\phi \pm 2k\pi,$$

so that

$$\theta = \frac{p}{q}\phi \pm \frac{2k\pi}{q}.$$

There are q sets of values of $\cos\theta$ and $\sin\theta$, corresponding to $k = 0, 1, 2 \ldots, q - 1$; and there are therefore q values of

$$(\cos \phi + i \sin \phi)^{\frac{p}{q}}.$$

Taking $k = 0$, we get $\theta = p\phi/q$, and

$$(\cos \phi + i \sin \phi)^{\frac{p}{q}} = \cos \frac{p}{q}\phi + i \sin \frac{p}{q}\phi,$$

as we wished to show.

★**103. The roots of a complex number.** — By a method similar to that just used, we shall now prove that, if k is any positive integer, any complex number (other than zero) has exactly k distinct kth roots. Let the number be

$$z = r (\cos \phi + i \sin \phi), \text{ where } r \neq 0.$$

We wish to show that z has k distinct kth roots. We shall try to determine values of ρ and θ so that the number $w = \rho (\cos \theta + i \sin \theta)$ shall be a kth root of z; that is, so that

$$w^k = \rho^k (\cos \theta + i \sin \theta)^k = r (\cos \phi + i \sin \phi).$$

By De Moivre's theorem, this is equivalent to

$$w^k = \rho^k (\cos k\theta + i \sin k\theta) = r (\cos \phi + i \sin \phi).$$

Separating reals and imaginaries, we have

(1) $$\rho^k \cos k\theta = r \cos \phi,$$

(2) $$\rho^k \sin k\theta = r \sin \phi.$$

Squaring and adding, we get

$$\rho^{2k}(\cos^2 k\theta + \sin^2 k\theta) = r^2(\cos^2\phi + \sin^2\phi),$$

so that $\rho^{2k} = r^2$ or $\rho = \sqrt[k]{r}$. This is the unique positive kth root of the positive number r. From equations (1) and (2) we then have $\cos k\theta = \cos\phi$ and $\sin k\theta = \sin\phi$. These equations are satisfied if $k\theta$ has any one of the following values:

$$k\theta = \phi, \phi + 2\pi, \phi + 4\pi, \cdots, \phi + 2(k-1)\pi.$$

or

$$\theta = \frac{\phi}{k}, \frac{\phi}{k} + \frac{2\pi}{k}, \frac{\phi}{k} + \frac{4\pi}{k}, \cdots, \frac{\phi}{k} + \frac{2(k-1)\pi}{k}.$$

Moreover, these values of θ yield distinct values of w:

$$w_1 = \sqrt[k]{r}\left(\cos\frac{\phi}{k} + i\sin\frac{\phi}{k}\right),$$

$$w_2 = \sqrt[k]{r}\left[\cos\left(\frac{\phi}{k} + \frac{2\pi}{k}\right) + i\sin\left(\frac{\phi}{k} + \frac{2\pi}{k}\right)\right],$$

. . . . . .

$$w_k = \sqrt[k]{r}\left[\cos\left(\frac{\phi}{k} + \frac{2\pi(k-1)}{k}\right) + i\sin\left(\frac{\phi}{k} + \frac{2\pi(k-1)}{k}\right)\right].$$

On expanding the kth power of any one of these numbers by De Moivre's theorem, we see that it yields

$$z = r(\cos\phi + i\sin\phi).$$

So these k distinct roots exist. There are no other roots; for example, if we should take $k\theta = \phi + 2k\pi$, or

$$\theta = \frac{\phi}{k} + 2\pi,$$

this would merely duplicate the value of w_1. The amplitude of w_1 is ϕ/k. The roots $w_1, w_2, \cdots, w_k$ are equally spaced on a circle

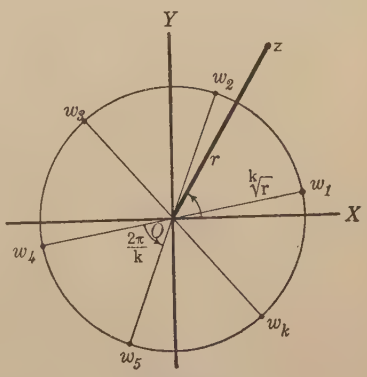

FIG. 78

of radius $\sqrt[k]{r}$ drawn about the origin, the difference between successive amplitudes being $2\pi/k$.

EXAMPLE 1. Find the three cube roots of unity. (Cube roots of 1.) Here

$$r(\cos\phi + i\sin\phi)^r = 1,$$

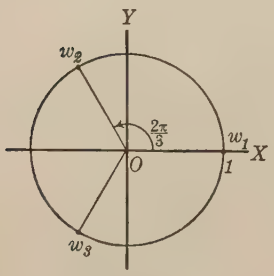

FIG. 79

so that $r = 1$ and $\phi = 0$. Then $\sqrt[3]{r} = 1$, the three values of θ are 0, $2\pi/3$, and $4\pi/3$, and the three roots are

$$w_1 = (\cos 0 + i\sin 0) = 1,$$

$$w_2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{i}{2}\sqrt{3},$$

$$w_3 = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$

EXAMPLE 2. Find the fifth roots of $z = 1 + i$. Writing this in polar form, we get

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

so that $r = \sqrt{2}$ and $\phi = \pi/4$. Then $\sqrt[5]{r} = \sqrt[10]{2} = 1.07$. The five values of θ are

$$\frac{\pi}{20}, \frac{\pi}{20} + \frac{2\pi}{5}, \frac{\pi}{20} + \frac{4\pi}{5}, \frac{\pi}{20} + \frac{6\pi}{5}, \frac{\pi}{20} + \frac{8\pi}{5}.$$

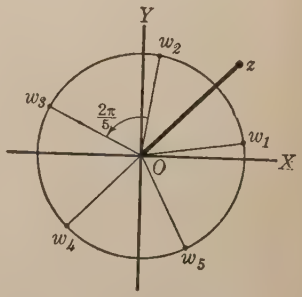

FIG. 80

The roots are

$$w_1 = \sqrt[10]{2}\left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right) = 1.06 + .167\,i.$$

$$w_2 = \sqrt[10]{2}\left(\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20}\right) = .167 + 1.06\,i.$$

$$w_3 = \sqrt[10]{2}\left(\cos\frac{17\pi}{20} + i\sin\frac{17\pi}{20}\right) = -.953 + .486\,i.$$

$$w_4 = \sqrt[10]{2}\left(\cos\frac{25\pi}{20} + i\sin\frac{25\pi}{20}\right) = -.756 - .756\,i.$$

$$w_5 = \sqrt[10]{2}\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right) = .486 - .953\,i.$$

EXAMPLE 3. Find the value of

$$z = \frac{625}{(3 - 4i)^4},$$

stating the results correctly to four significant figures.

We can take

$$z = 625\,(3 - 4i)^{-4} = 625\,[5\,(\cos\phi + i\sin\phi)]^{-4},$$

where $\phi = -\arccos 3/5 = -53° 7.8'$.

Therefore, by De Moivre's theorem,

$$\begin{aligned} z &= \frac{625}{5^4}\,[\cos(-4\phi) + i\sin(-4\phi)] \\ &= \cos 212° 31.2' - i\sin 212° 31.2' \\ &= -.84321 + .53759\,i. \end{aligned}$$

EXERCISES

1. Find the fourth roots of 5. *Ans.* $\sqrt{5}, \sqrt{5}i, -\sqrt{5}, -\sqrt{5}i.$
2. Find the sixth roots of 27.
3. Find the fifth roots of 32.
   Ans. $2, .618 \pm 1.902\,i, -1.618 \pm 1.176\,i.$
4. Find the seventh roots of 4.
5. Find the cube roots of $2 - i$.

Ans. $(1.2921 - .2013i), (-.4717 + 1.2196i), (-.8204 - 1.0183i).$

6. Find the fifth roots of $(-3 + 4i)$.
7. Find the value of $(2 - 6i)^6$ and plot the point which represents it. *Ans.* $(22526 - 59904i).$

8. Find the value of $(-3 + 2i)^5$, and plot the point which represents it.

9. Find the value of $\dfrac{1000}{(-8 - 6i)^3}$. *Ans.* $.352 + .936i$.

10. Find the value of $\dfrac{4913}{(-8 + 15i)^3}$.

✶**104. Definition of the number e.** — Let us compute the value of the number

$$S_n = \left(1 + \frac{1}{n}\right)^n$$

for various integral values of n. We have

$$\begin{aligned}
S_1 &= (1 + 1)^1 = 2,\\
S_2 &= (1 + \tfrac{1}{2})^2 = \tfrac{9}{4} = 2.25,\\
S_3 &= (1 + \tfrac{1}{3})^3 = \tfrac{6}{2}\tfrac{4}{7} = 2.37,\\
S_4 &= (1 + \tfrac{1}{4})^4 = \tfrac{6}{2}\tfrac{2}{5}\tfrac{5}{6} = 2.44,\\
S_{10} &= (1 + \tfrac{1}{10})^{10} = 2.59,\\
S_{100} &= (1 + \tfrac{1}{100})^{100} = 2.70,\\
S_{1000} &= 2.716.
\end{aligned}$$

It will be noticed that when n increases, the value of S_n increases. But S_n does not increase indefinitely. In fact it can be shown that, as n increases indefinitely, the value of S_n approaches a limit; that is, there is a certain number the difference between which and S_n becomes (and remains) as small as we wish, if the value of n is taken sufficiently large. This limit we denote by the letter e, so that

$$e = \lim_{n=\infty} \left(1 + \frac{1}{n}\right)^n.$$

This is read: e is the limit as n becomes infinite of $\left(1 + \dfrac{1}{n}\right)^n$. This number e is very important in higher mathematics. To six significant figures its value is $e = 2.71828$.

★105. A series expression for e^x. — The quantity e^x is a function of x which may be defined in the following way. Evidently

$$e^x = [\lim_{n=\infty} S_n]^x = \left[\lim_{n=\infty}\left(1+\frac{1}{n}\right)^n\right]^x.$$

It can be shown that this is equal to

$$e^x = \lim_{n=\infty}\left[\left(1+\frac{1}{n}\right)^{nx}\right],$$

and by expanding the expression in brackets by the binomial theorem and then taking the limit, we can reduce this to

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

where $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3$, $4! = 1 \cdot 2 \cdot 3 \cdot 4$, etc. It is shown in calculus that if the sum of the first n terms of this series be formed for any given value of x, this sum really approaches as a limit (or "converges to") the corresponding value of e^x when the number of terms, n, increases without limit.

EXAMPLE. From this series we can compute to any desired number of places the value of e^x for any value of x. Let us, for example, take $x = 1$, and so compute the value of e:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

$1 = 1.$
$1 = 1.$
$1/2! = 0.5$
$1/3! = 0.166667$
$1/4! = 0.041667$
$1/5! = 0.008333$
$1/6! = 0.001389$
$1/7! = 0.000198$
$1/8! = 0.000025$
$1/9! = 0.000003$
Sum $= 2.718282$; $e = 2.71828$ to six figures.

EXERCISE

Compute, from the series, the value of e^2 to 4 decimal places, and the value of $\sqrt{e}$ to 6 decimal places.

***106. Computation of sines and cosines by series.** — In a course in calculus the following series developments are obtained for $\sin x$ and $\cos x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

These series give the correct values of the sine and cosine for any value of x. When substituting we must express x in radians. Assuming these expressions, we can at once note many of the properties of the functions. For example, $\sin x$ is an odd function, since in its expansion only odd powers of x occur, and, if x changes sign, every term changes sign. In a similar way we see that $\cos x$ is an even function. Also, if x is numerically small, $\sin x$ is very nearly equal to x, since for small values of x the higher powers of x are very small indeed, so that all terms after the first are very small.

The series are useful in forming tables of the trigonometric functions, for the series converge rapidly for small values of x; that is, only a few terms are necessary to give the value of the function with considerable accuracy.

EXAMPLE. Compute $\sin 10°$ and $\cos 10°$ to five significant figures. We write

$$10° = \frac{180°}{18} = \frac{\pi}{18} \text{ radians} = .174533 \text{ radians},$$

so that $x = .174533$. Therefore

$$\begin{aligned}\sin 10° &= .174533 - \frac{(.174533)^3}{6} + \frac{(.174533)^5}{120} - \cdots \\ &= .174533 - .000876 + .000001 - \cdots \\ &= .17365.\end{aligned}$$

$$\cos 10° = 1 - \frac{(.174533)^2}{2} + \frac{(.174533)^4}{24} - \frac{(.174533)^6}{720} + \cdots$$
$$= 1 - .015230 + .000039 + \cdots$$
$$= .98481.$$

From these values we can easily compute others, as, for example,

$$\sin 20° = 2 \sin 10° \cos 10° = 2 \times .17365 \times .98481 = .34202.$$

EXERCISES

Compute the sine and cosine of each of the following angles to five figures, and verify the results by means of a table.

1. 2°. **2.** 7° 30′. **3.** 15°. **4.** 5°.

***107. Relations between trigonometric and exponential functions.** — In Articles **105** and **106** we have had the following expansions:

(1) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

(2) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

(3) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

These series are valid for both real and complex values of x; that is, they give the correct values of the functions for real values of x, and may be used to define the functions for complex values of x. In place of x in (1) let us substitute the imaginary number xi, where $i = \sqrt{-1}$, and separate the real and imaginary terms.

$$e^{xi} = 1 + xi + \frac{x^2 i^2}{2!} + \frac{x^3 i^3}{3!} + \frac{x^4 i^4}{4!} + \cdots$$

or

$$e^{xi} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right).$$

Therefore, by (2) and (3),

$$e^{xi} = \cos x + i \sin x. \tag{4}$$

We see from this that the modulus of e^{xi} is equal to 1, and the amplitude of e^{xi} is equal to x for any real value of x. Evidently

$$e^{x+iy} = e^x \cdot e^{yi} = e^x (\cos y + i \sin y),$$

so that the absolute value of e^{x+iy} is equal to e^x and the amplitude is the coefficient of the imaginary part of the exponent, or y.

From the equation

$$e^{xi} = \cos x + i \sin x,$$

we get

$$e^{-xi} = \cos x - i \sin x,$$

by substituting $-x$ for x. If we subtract the second of these equations from the first, we find

$$e^{xi} - e^{-xi} = 2i \sin x$$

or

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i}. \tag{5}$$

After adding the equations, we see that

$$\cos x = \frac{e^{xi} + e^{-xi}}{2}. \tag{6}$$

From these equations it is possible to prove many trigonometric relations. For example, let us show that

$$\sin^2 x + \cos^2 x = 1.$$

From (4) and (5),

$$\sin^2 x + \cos^2 x = \frac{e^{2xi} - 2 + e^{-2xi}}{4i^2} + \frac{e^{2xi} + 2 + e^{-2xi}}{4}$$
$$= \frac{e^{2xi} - 2 + e^{-2xi}}{-4} + \frac{e^{2xi} + 2 + e^{-2xi}}{4} = 1.$$

EXERCISES

1. Show that $\sin^2 x + \cos^2 x = 1$ by multiplying the expressions for e^{xi} and e^{-xi}.

By means of equations (5) and (6) prove the following relations.

2. $\sin 2x = 2 \sin x \cos x$.
3. $\cos 2x = \cos^2 x - \sin^2 x$.
4. $\sin (x + y) = \sin x \cos y + \cos x \sin y$.
5. $\cos (x + y) = \cos x \cos y - \sin x \sin y$.
6. $\sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$.
7. $\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$.

★**108. Computation of logarithms by series.** — Either by starting with the expansion of Article **105** or by working entirely by the methods of calculus, it is possible to find the following series:

$$(1) \qquad \log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

In this logarithm the base is the number e defined in Article **104**, and the series gives the correct value of the logarithm for any real value of x between -1 and 1.

Substituting $-x$ for x in this series, we get

$$(2) \qquad \log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots.$$

Subtracting this from the series for $\log_e (1 + x)$, we get

$$(3) \qquad \log_e \left(\frac{1 + x}{1 - x}\right) = \log_e (1 + x) - \log_e (1 - x)$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

This series is more useful than the series (1) for computing logarithms, because it converges much more rapidly. That is, fewer terms are required to give a satisfactory result with (3) than with (1).

EXAMPLE 1. Compute $\log_e 2$.

Here let us set

$$\frac{1+x}{1-x} = 2,$$

which we can solve for x, getting $x = 1/3$. Then

$$\log_e 2 = \log_e\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = 2\left(\frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} + \cdots\right).$$

$$\begin{aligned}
\tfrac{1}{3} &= .333333\\
(\tfrac{1}{3})^3/3 &= .012345\\
(\tfrac{1}{3})^5/5 &= .000823\\
(\tfrac{1}{3})^7/7 &= .000065\\
(\tfrac{1}{3})^9/9 &= .000006\\
\tfrac{1}{2}\log_e 2 &= .34657\\
\log_e 2 &= .6931.
\end{aligned}$$

Series (3) can be written so as to give us $\log_e (n+1)$ when $\log_e n$ is known. Set

$$\frac{n+1}{n} = \frac{1+x}{1-x}.$$

or

$$n + 1 - nx - x = n + nx,$$

from which we get

$$x = \frac{1}{2n+1}.$$

Using this value of x in (3), we get

$$\log_e\left(\frac{n+1}{n}\right) = 2\left[\frac{1}{(2n+1)} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right]$$

and

(4) $$\log_e (n+1) = \log_e n + 2\left[\frac{1}{(2n+1)} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right].$$

EXAMPLE 2. Find $\log_e 3$, given $\log_e 2 = .69315$.
From (4), using $n = 2$, we get

$$\begin{aligned}\log_e 3 &= \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \cdots\right)\\ &= .69315 + 2(.2 + .002667 + .000064 + \cdots)\\ &= 1.09861.\end{aligned}$$

EXAMPLE 3. Find $\log_e 10$, given $\log_e 3 = 1.09861$.
We can write

$$\log_e 10 = \log_e (9 + 1).$$

We can find $\log_e 9 = \log_e 3^2 = 2 \times 1.09861$, and in (4) take $n = 9$. Then

$$\begin{aligned}\log_e 10 &= 2 \times 1.09861 + 2\left(\frac{1}{19} + \frac{1}{3.19^3} + \frac{1}{5.19^5} + \cdots\right)\\ &= 2.19722 + 2(.052632 + .000051 + \cdots)\\ &= 2.30259.\end{aligned}$$

★**109. Formation of logarithmic tables.** — By using series (3) and (4) of Article **108**, we can form tables of logarithms to the base e. From these it is possible to form tables of logarithms to the base 10 by means of the relation

$$\log_{10} M = \frac{\log_e M}{\log_e 10}.$$

To prove this relation, set

$$x = \log_{10} M, \quad y = \log_e M,$$

so that

$$10^x = M, \qquad e^y = M.$$

Therefore

$$10^x = e^y$$

and

$$e^{\frac{y}{x}} = 10.$$

Therefore $\log_e 10 = \dfrac{y}{x}$, by the definition of logarithms,

or

$$x = \frac{y}{\log_e 10}.$$

This gives the formula

$$\log_{10} M = \frac{\log_e M}{\log_e 10}.$$

Using the value $\log_e 10 = 2.30259$, which we computed in Article 108, we get

$$\log_{10} M = \frac{\log_e M}{2.30259} = .43429 \log_e M.$$

EXAMPLE. Find $\log_{10} 2$, given $\log_e 2 = .69315$.
We have

$$\log_{10} 2 = .43429 \times .69315 = .30103.$$

EXERCISES

Compute the logarithms to the bases e and 10 of the integers from 1 to 20 to five places, and verify by means of a table.